The $\mathcal{A} - \mathcal{T}$ plane: a new tool for time series analysis

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Enigmas of Chance
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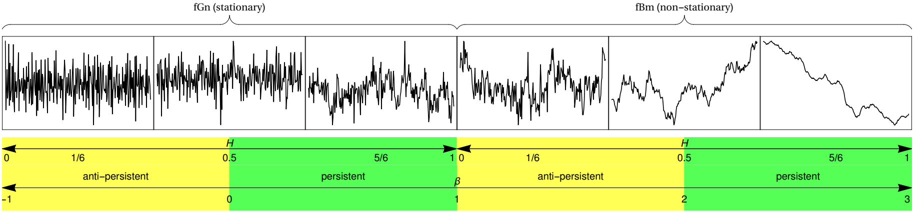
Plan

1. $H, \mathcal{A}, \mathcal{T}$
2. Analytical description
3. Applications
4. Relations to chaos
5. Extensions

Hurst exponent, H

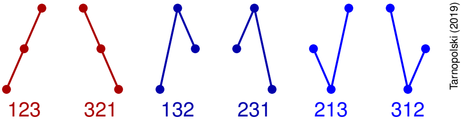
$$x(t) \doteq \lambda^{-H} x(\lambda t) \qquad \rho_k \propto |k|^{-\delta} \equiv |k|^{-(2-2H)}$$



The properties of H:

1. $0 < H < 1$,
2. $H = 1/2$ for an uncorrelated process,
3. $H > 1/2$ for a persistent (long-term memory, correlated) process,
4. $H < 1/2$ for an anti-persistent (short-term memory, anti-correlated) process.

1. Fraction of turning points, $\mathcal{T} = T/N$:



For white noise $\mathcal{T} = {}^{2}\!/_{3}$.

The $\mathcal{A}-\mathcal{T}$ plane

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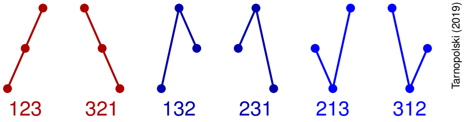


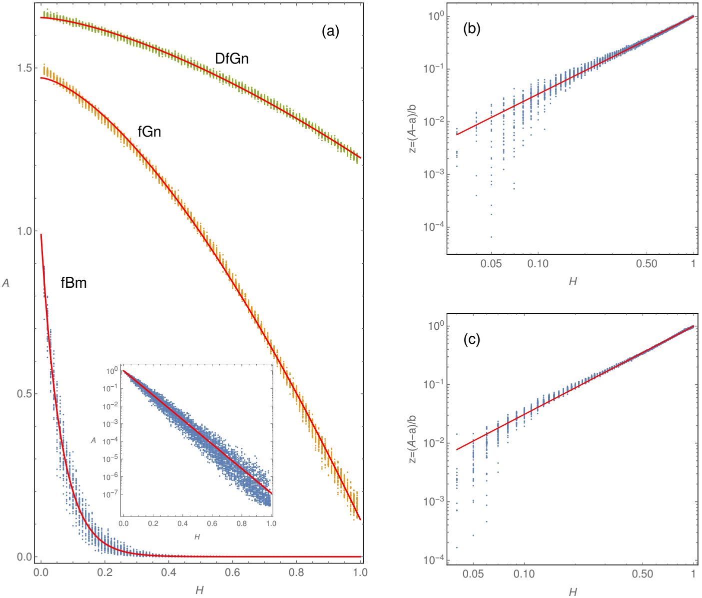
For white noise $\mathcal{T} = {}^2/_3$.

2. Abbe value:

$$\mathcal{A} = \frac{\frac{1}{N-1}\sum\limits_{k=1}^{N-1}(x_{k+1}-x_k)^2}{\frac{2}{N}\sum\limits_{k=1}^{N}(x_k-\bar{x})^2} = \frac{1}{2}\frac{\mathrm{var}\,(dX)}{\mathrm{var}\,(X)}$$

Normalised so that $\mathcal{A} = 1$ for white noise.

The $\mathcal{A} - \mathcal{T}$ plane



Tarnopolski (2016)

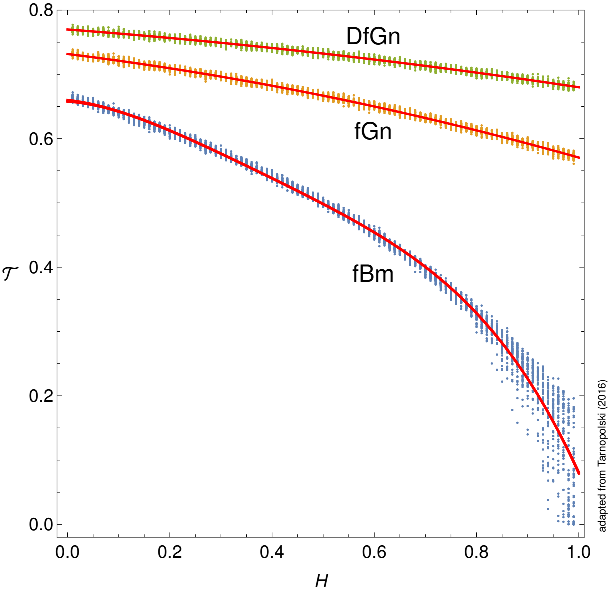
The $\mathcal{A} - \mathcal{T}$ plane

Parameters of the fits of the relation $\mathcal{A}(H)$ for $n = 2^{14}$. The errors in brackets correspond to the last significant digit.

Process	Formula	a	b	c
fBm	$\mathcal{A}(H) = ae^{bH}$	0.989(2)	$-15.95(5)$	–
fGn	$\mathcal{A}(H) = a + bH^c$	1.4695(5)	$-1.3550(7)$	1.508(2)
DfGn	$\mathcal{A}(H) = a + bH^c$	1.6546(3)	$-0.4300(3)$	1.474(3)

Tarnopolski (2016)

The $\mathcal{A} - \mathcal{T}$ plane



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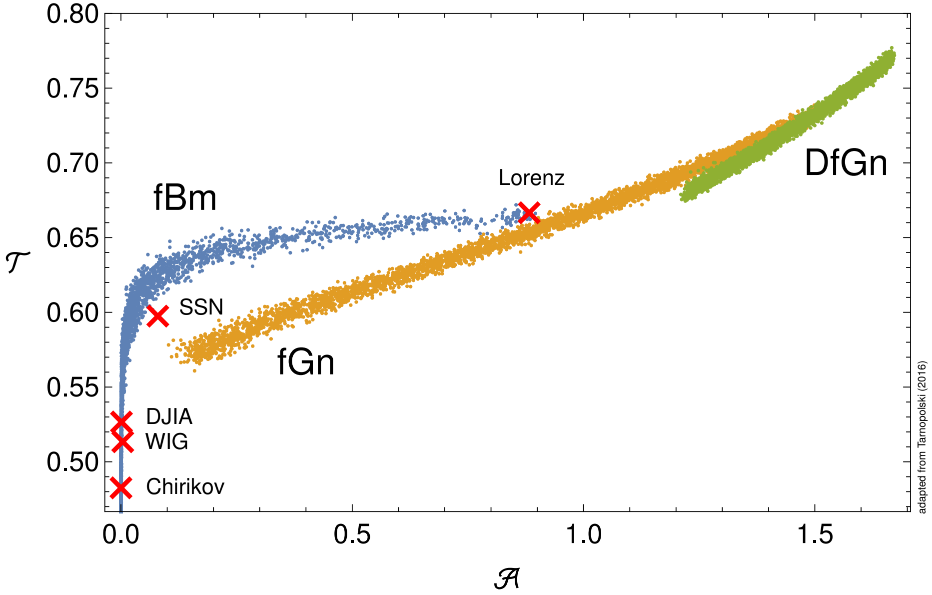
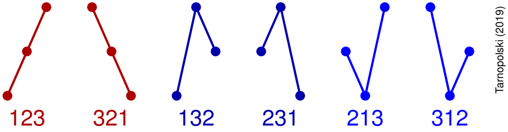
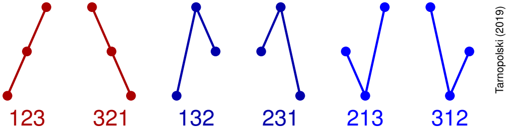




Consider three values x_i, x_{i+1}, x_{i+2} of a time series $\{x_t\}_{t=1}^{N}$.
Denote the probability of encountering a pattern π_p by p_{π_p}.

Analytical description



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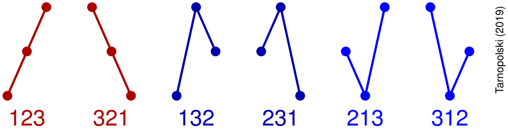
Theorem (Bandt & Shiha 2007)

For a Gaussian process X_t with stationary increments, $p_{123} = p_{321} = \alpha/2$, and the other patterns yield probability $(1-\alpha)/4$.

Analytical description



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$$p_{123} = \frac{1}{\pi} \arcsin \sqrt{\frac{1+\rho(1)}{2}} = \frac{1}{\pi} \arcsin \left(\frac{1}{2} \sqrt{\frac{1-\rho_2}{1-\rho_1}} \right)$$

Analytical description

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$$\mathcal{T}_{\mathrm{fBm}} = 1 - \frac{2}{\pi}\arcsin\left(2^{H-1}\right)$$

$$\mathcal{T}_{\mathrm{fGn}} = 1 - \frac{2}{\pi}\arcsin\left(\frac{1}{2}\sqrt{\frac{3^{2H} - 2^{2H+1} - 1}{2^{2H} - 4}}\right)$$

$$\mathcal{T}_{\mathrm{DfGn}} = 1 - \frac{2}{\pi}\arcsin\left(\frac{1}{2}\sqrt{\frac{(2^{2H}+2)^2 - (2\cdot 3^H)^2}{3^{2H} - 3\cdot 2^{2H+1} + 15}}\right)$$

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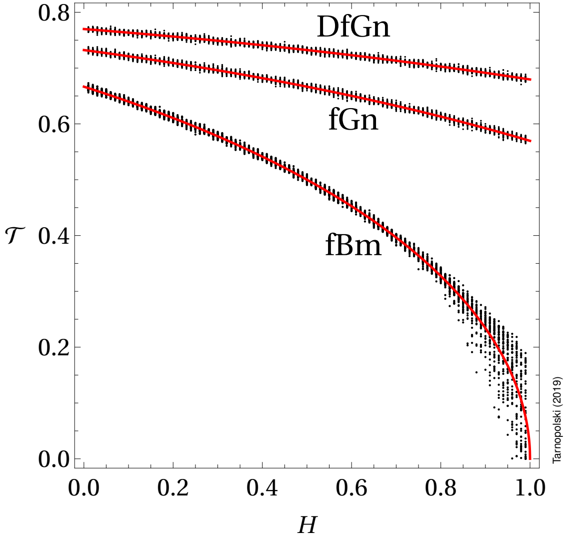
Higher order delays

x_i, x_{i+d}, x_{i+2d} — consecutive points for $d = 1$

Analytical description



Analytical description

$$\mathcal{A} = \frac{1}{2}\frac{\mathrm{var}\left(dX\right)}{\mathrm{var}\left(X\right)}$$

$$\mathcal{A}_{\mathrm{fBm}}(H,n) = \frac{1}{2}\frac{\mathrm{var}\left(G_{n-1}^{H}\right)}{\mathrm{var}\left(B_{n}^{H}\right)}$$

$$\mathcal{A}_{\mathrm{fGn}}(H,n) = \frac{1}{2}\frac{\mathrm{var}\left(Y_{n-1}^{H}\right)}{\mathrm{var}\left(G_{n}^{H}\right)}$$

Analytical description

$$\mathrm{var}\left(G_n^H\right) = \frac{1}{n-1} E\left[\sum_{j=0}^{n-1}\left(G_j - \frac{\sum_{k=0}^{n-1} G_k}{n}\right)^2\right]$$

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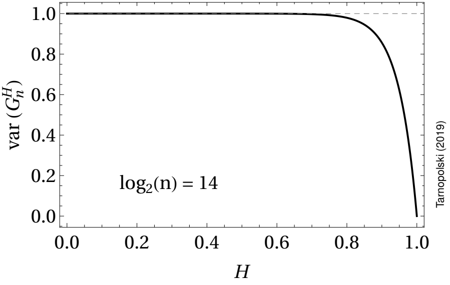

Analytical description

$$\operatorname{var}\left(Y_n^H\right) = \frac{2(2n^2-1) - n^2 2^{2H} + (n-1)^{2H} - 2n^{2H} + (n+1)^{2H}}{n(n-1)}$$

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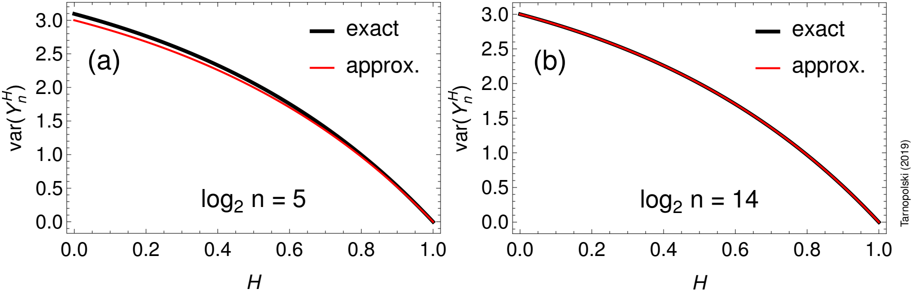

Analytical description

Delignières (2015):

$$\mathrm{var}\left(B_n^H\right) = \frac{1}{n(n-1)} \sum_{i=1}^{n-1} (n-i) i^{2H}$$

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$$\begin{aligned}\sum_{i=1}^{n-1} (n-i) i^{2H} =& \zeta(-2H-1, n) - n\zeta(-2H, n) + \\ & n\zeta(-2H) - \zeta(-2H-1)\end{aligned}$$

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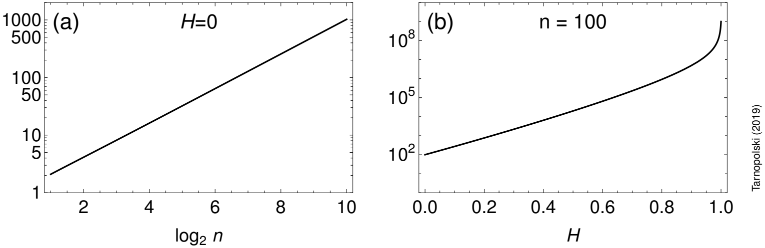

Analytical description

Hasse (1930) representation:

$$\zeta(s,n) = \frac{1}{s-1}\sum_{i=0}^{\infty}\frac{1}{i+1}\sum_{k=0}^{i}(-1)^k\binom{i}{k}(n+k)^{1-s}$$

valid for $s \neq 1$, $n > 0$.

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$$\mathrm{var}\left(B_n^H\right) \approx \frac{n^{2H}}{2(H+1)(2H+1)}$$

Analytical description

Working approximations

$$\begin{cases} \mathcal{A}_{\mathrm{fBm}}(H,n) & = (H+1)(2H+1)n^{-2H} \\ \mathcal{T}_{\mathrm{fBm}}(H) & = 1 - \frac{2}{\pi}\arcsin\left(2^{H-1}\right) \end{cases}$$

and

$$\begin{cases} \mathcal{A}_{\mathrm{fGn}}(H) = 2 - 2^{2H-1} \\ \mathcal{T}_{\mathrm{fGn}}(H) = 1 - \frac{2}{\pi}\arcsin\left(\frac{1}{2}\sqrt{\frac{3^{2H}-2^{2H+1}-1}{2^{2H}-4}}\right) \end{cases}$$

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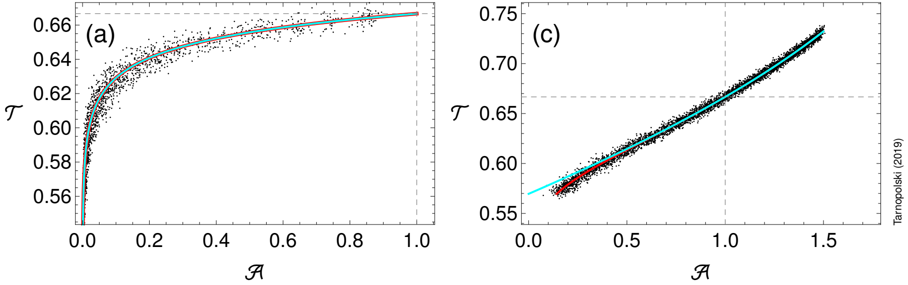
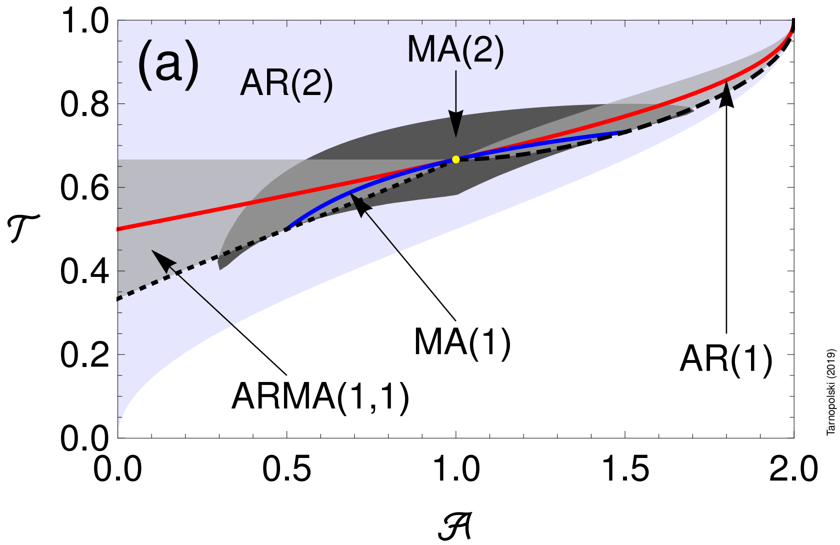
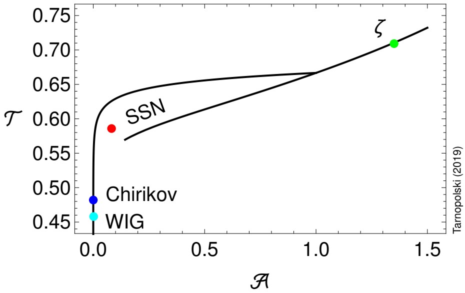
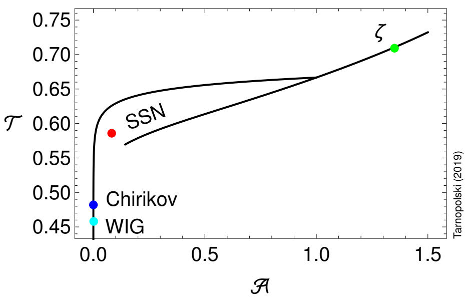


Analytical description—ARMA processes



Examples

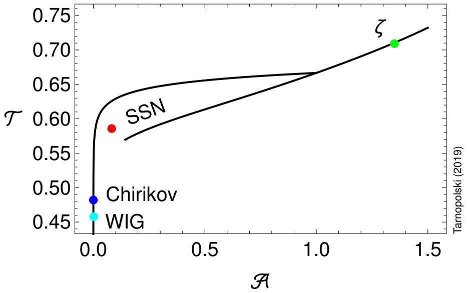

0.75
0.70
0.65
0.60
0.55
0.50
0.45
$\mathcal{T}$
0.0
0.5
1.0
1.5
$\mathcal{A}$
ζ
SSN
Chirikov
WIG
Tarnopolski (2019)



- WIG:

$$H_{\mathcal{A}-\mathcal{T}} = 0.49\text{–}0.59 \quad H_{\rm wav} = 0.48$$

Examples



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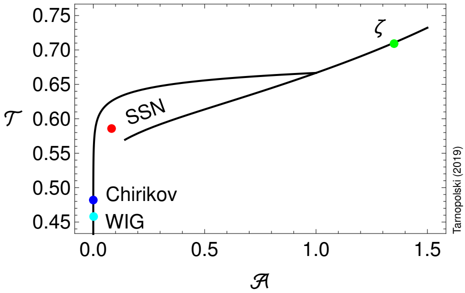
$$H_{\mathcal{A}-\mathcal{T}} \approx 0.5 \quad H_{\mathrm{wav}} = 0.48$$

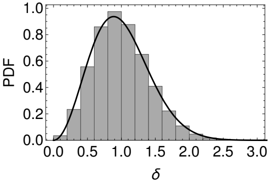


- zeros of Riemann zeta:

$$1/2 + i\gamma_n$$

$$\delta_n = \frac{\gamma_{n+1} - \gamma_n}{2\pi} \ln\left(\frac{\gamma_n}{2\pi}\right)$$

$$H_{\mathcal{A}-\mathcal{T}} = 0.19 \quad H_{\mathrm{wav}} = 0.06$$



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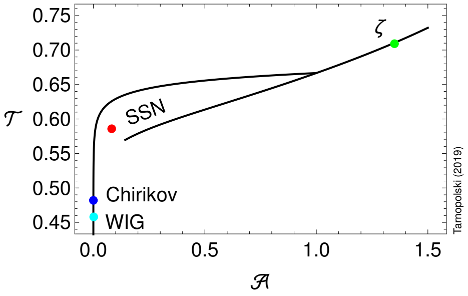

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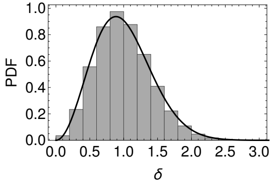
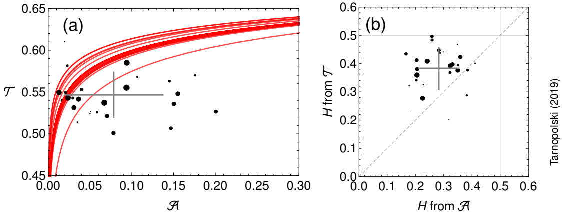


- SSN:

$$H_{\mathcal{A}-\mathcal{T}} = 0.18\text{–}0.27 \quad H_{\rm wav} = 0.3$$

Examples

mRNA molecules in live *E. coli* (Golding & Cox 2006):



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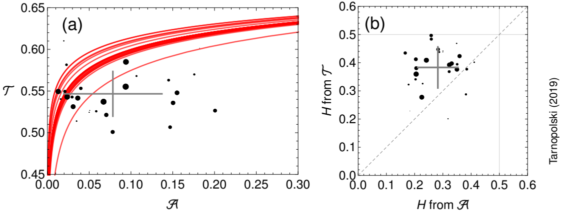
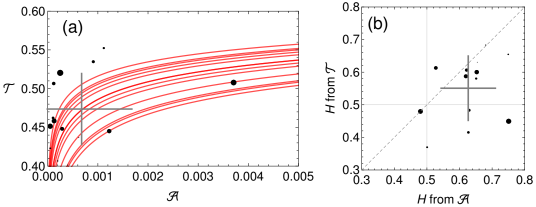
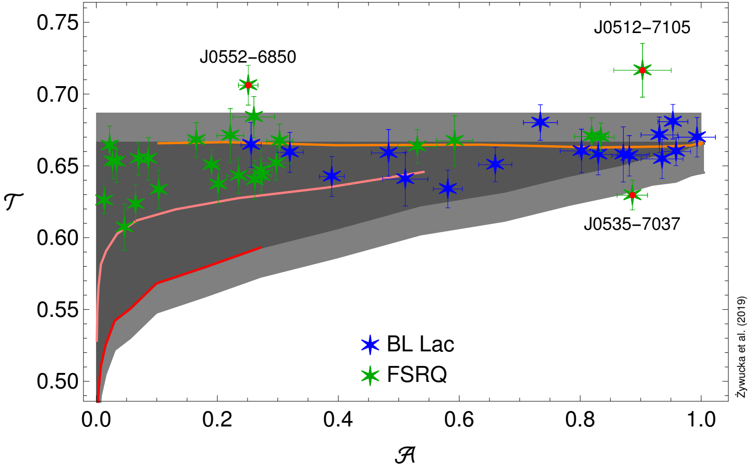


Amoeboid motion (Makarava et al. 2014):

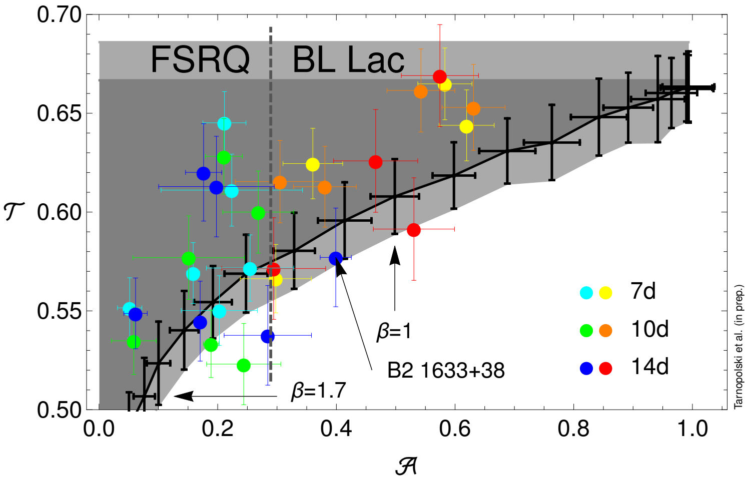


Examples—active galactic nuclei (AGN)

$1/f^{\beta} + C$

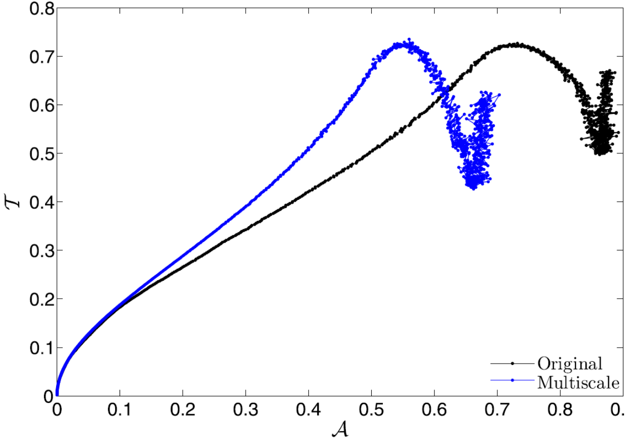


$\langle\mathcal{A}\rangle_{\rm FSRQ} = 0.29 \pm 0.05$ and $\langle\mathcal{A}\rangle_{\rm BLLac} = 0.71 \pm 0.06$

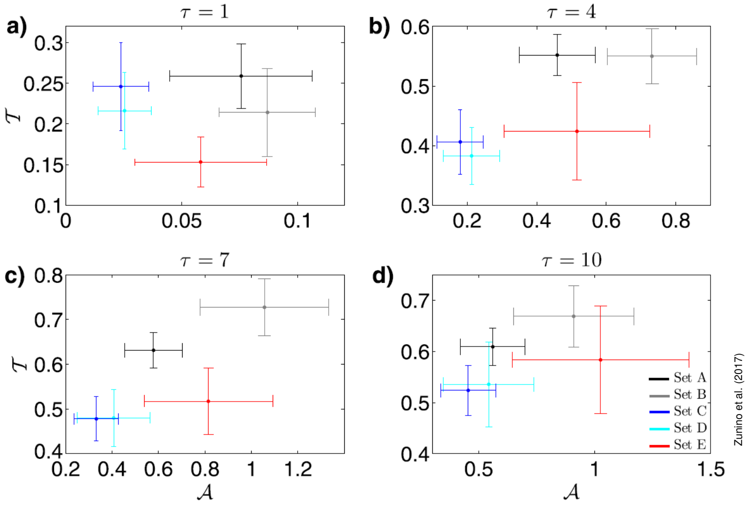

0.70
0.65
0.60
0.55
0.50
$\mathcal{T}$
0.0
0.2
0.4
0.6
0.8
1.0
$\mathcal{A}$
FSRQ
BL Lac
β=1
β=1.7
B2 1633+38
7d
10d
14d
Tarnopolski et al. (in prep.)

Extension—coarse graining

$$y_j^\tau = \frac{1}{\tau} \sum_{k=(j-1)\tau+1}^{j\tau} x_k \text{ for } j \in \{1, \ldots, \lfloor N/\tau \rfloor\}$$

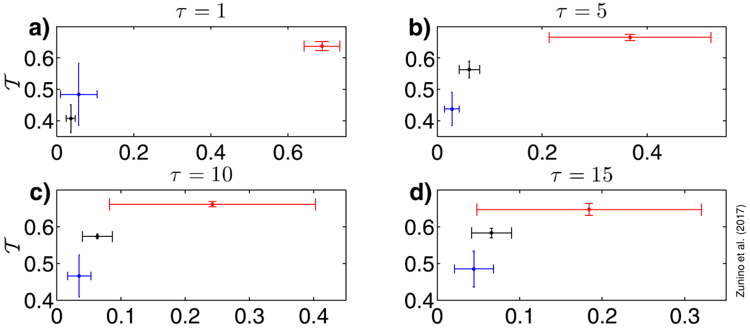


Zunino et al. (2017)



A: scalp healthy–open eyes; B: scalp healthy–closed eyes; C: intracranial epileptic—in seizure region; D: intracranial epileptic—outside seizure region; E: intracranial—during seizure.

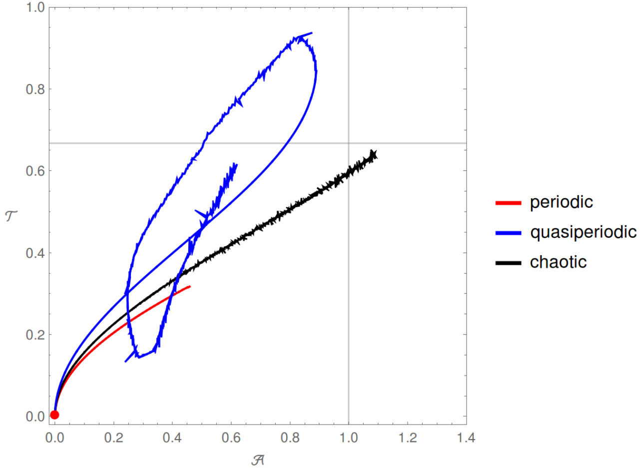
Extension—coarse graining



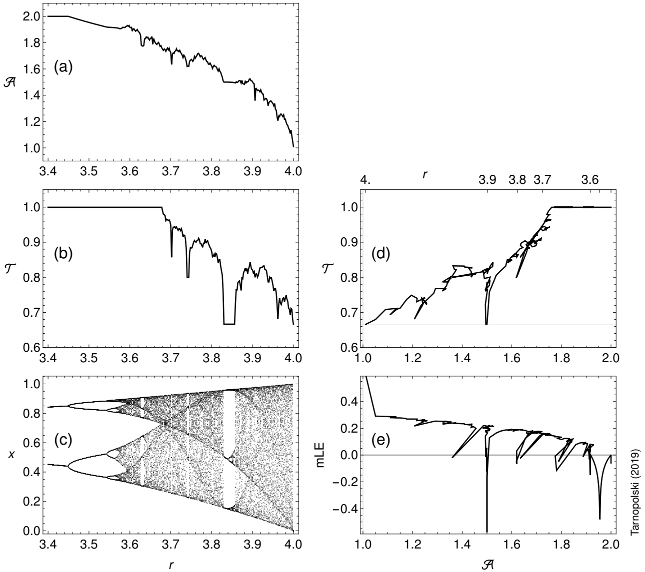
NSR—normal sinus rhytm; CHF—congestive heart failure; AF—atrial fibrillation.

Extension—coarse graining—sinusoidally driven thermostat

$$\Gamma'(t) = u^2(t) - 1, \quad u'(t) = A\sin(\omega t) - \Gamma(t)u(t)$$

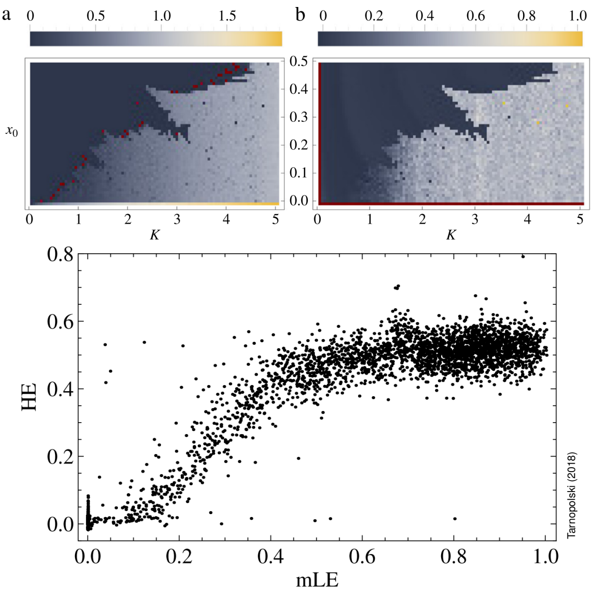


based on Zhao & Morales (2018)



(a)
$\mathcal{A}$
(b)
$\mathcal{T}$
(c)
x
r
(d)
r
4.
3.9
3.8
3.7
3.6
(e)
mLE
$\mathcal{A}$
Tarnopolski (2019)

Chaos—Chirikov map



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- Order patterns of higher delays;
- Can the Abbe value be generalised?

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- A robust methodology for time series classification;
- Order patterns of higher delays;
- Can the Abbe value be generalised?
- Other...

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Thank you for your attention!