International Journal of Forecasting 32 (2016) 398–410

Contents lists available at ScienceDirect

International Journal of Forecasting

journal homepage: www.elsevier.com/locate/ijforecast

Low and high prices can improve volatility forecasts during periods of turmoil

Piotr Fiszeder*, Grzegorz Perczak

Nicolaus Copernicus University, Faculty of Economic Sciences and Management, Department of Econometrics and Statistics, ul. Gagarina 13a, 87-100 Toruń, Poland

ARTICLE INFO

Keywords:
GARCH model
Low and high prices
NIG distribution
Turmoil period
Volatility forecasting

ABSTRACT

In this study, we describe a modification of the GARCH model that we have formulated, where its parameters are estimated based on closing prices as well as on information related to daily minimum and maximum prices. In an empirical application, we show that the use of low and high prices in the derivation of the likelihood function of the GARCH model improved the volatility estimation and increased the accuracy of volatility forecasts based on this model during the period of turmoil, relative to using closing prices only. This analysis was performed for two stock indices from developed markets, i.e., S&P 500 and FTSE 100, and for two stock indices from emerging markets, i.e., the Polish WIG20 index and the Greek Athex Composite Share Price Index. The main result obtained in this study is robust to both the forecast evaluation criterion applied and the proxy used for the daily volatility.

1. Introduction

The modelling and forecasting of the volatility of asset returns is a key issue in many financial and economic applications. One of the most popular volatility models is the GARCH model, and the estimation of its parameters is based solely on the daily closing prices in the majority of cases. However, a single return gives a weak signal for the current level of volatility. This implies that GARCH models are poorly suited to situations where the volatility changes suddenly to a new level. For instance, when the volatility increases sharply on day t and subsequent days, the conditional variance of the GARCH model will not change on day t and will increase only gradually on the subsequent days. Thus, the conditional variance will take many periods to reach a new level of volatility (e.g. Andersen, Bollerslev, Diebold, & Labys, 2003; Hansen, Huang, & Shek, 2012). However, while the commonly available databases do contain the daily closing prices, they also include daily low and high prices, which can be used successfully for volatility estimation. The use of low and high prices is one area in which extensive research, both theoretical and empirical, is currently being conducted (see the review by Chou, Chou, & Liu, 2010). This renewed interest within the scientific community is mainly because the application of such data yields more accurate estimates and forecasts of volatility than those based only on closing prices (e.g., Chou, 2005; Li & Hong, 2011).

The research concerning the use of data on low and high prices can be divided into three main groups (we deliberately omit the use of so-called intraday or high frequency data, and focus on daily data). The first group consists of the so-called price range estimators, which include the best-known estimators of Garman and Klass (1980), Parkinson (1980), Rogers and Satchell (1991) and Yang and Zhang (2000). Financial market practitioners commonly use range estimators to estimate the volatility

* Corresponding author.
E-mail addresses: piotr.fiszeder@umk.pl (P. Fiszeder), grzegorz.perczak@gmail.com (G. Perczak).

http://dx.doi.org/10.1016/j.ijforecast.2015.07.003
0169-2070/

because they are significantly more efficient than the estimator calculated as the daily squared return of closing prices. However, these estimators are less popular among scientists because they neglect the temporal dependence of returns (such as conditional heteroscedasticity). The second group are the so-called range-based volatility models, which are widely used for closing prices or their modifications, although they are applied directly to the modelling of the price range (e.g., see Alizadeh, Brandt, & Diebold, 2002; Brandt & Jones, 2006; Chou, 2005; Engle & Gallo, 2006). The third group are models of interval-valued data (e.g., see Arroyo, Espínola, & Maté, 2011; Arroyo, González-Rivera, & Maté, 2010; Maia & de Carvalho, 2011; Maia, de Carvalho, & Ludermir, 2008). High-low intervals are linked naturally to the concept of volatility. This study does not discuss the latter two uses of low and high prices.

Only a few studies have used low and high prices directly for formulating the estimation procedure in existing and known volatility models. These include the GARCH models of Lildholdt (2002) and Venter, De Jongh, and Griebenow (2005), who derived likelihood functions based on low, high and closing prices. The present study makes two main contributions. The first is the presentation of a modification of the GARCH model, where the parameters are estimated based on low, high and closing prices (for details and less complex parameterizations, see Perczak & Fiszeder, 2014). Lildholdt (2002) assumed that, over each day, a new incremental log-price process follows an arithmetic Brownian motion with a constant volatility for that day, and therefore applied the GARCH model with a normal conditional innovation distribution. However, it is well known that the normal distribution is often too light-tailed to be an appropriate distribution for most financial time series. Therefore, similarly to Venter et al. (2005), we assume a normal-inverse Gaussian (NIG) conditional innovation distribution for the GARCH model. Furthermore, our formulation of the model differs in two respects. First, we apply the significantly more efficient range estimator of the variance, instead of the estimator calculated as the daily squared return of closing prices, as is commonly used in the standard GARCH model. Second, we assume slight simplifications where different parameterizations of random variables and stochastic processes are applied.

The study's second main contribution is to show that the use of additional information related to low and high prices in the derivation of the likelihood function of the GARCH model can improve the volatility estimation and increase the accuracy of volatility forecasts based on a model for periods of turmoil, compared with only applying closing prices. The idea of periods of turmoil refers to periods with large declines in stock prices and very high levels of volatility. To the best of our knowledge, this is the first attempt in the literature to demonstrate the superiority of this approach for forecasting. This issue is important from a practical viewpoint, because low and high prices are almost always commonly available with closing prices for financial series. Therefore, it can be stated that the omission of such data leads to the loss of important information.

It is well known that the extreme values that are associated with turbulent and crisis periods have a significant influence on the estimation results. One of the main weaknesses of the GARCH process where the parameters are estimated based on closing prices is a slow response to abrupt changes in the market (e.g. Andersen et al., 2003; Hansen et al., 2012). The use of low and high prices in the estimation of the parameters should reduce the impact of this negative effect significantly.

The remainder of this paper is organized as follows. Section 2 provides definitions of the distributions and processes employed in this study. Section 3 describes the parameterization of the GARCH model, where the parameters are estimated based on low, high and closing prices. In Section 4, this approach is then used to model the volatility of two well-known stock indices from developed markets, S&P 500 and FTSE 100, as well as the Polish stock index WIG20. Section 5 verifies the forecasting performance both for the usual period and for the period of turmoil due to the financial crisis in the USA. In Section 6, we perform a robustness check for additional proxies for volatility and for a different period of turmoil, namely the Greek debt crisis, using the Athex Composite Share Price Index. Section 7 provides our conclusions.

2. Definitions of the distributions and processes employed in this study

Let $S_{t,\tau}$ be the price of a financial instrument observed on day t ($t \in N, 0 < t$) after time τ ($0 \le \tau \le 1$) from the last quotation the day before. Thus, there is the identity $S_{t-1,1} = S_{t,0}$. The daily (24-hour) minimum and maximum prices are defined as $L_t = \min_{0\le\tau\le1} S_{t,\tau}$ and $H_t = \max_{0\le\tau\le1} S_{t,\tau}$, respectively. In addition, we employ the following definitions of daily low, high and closing returns: $A_t = \ln\left(L_t/S_{t,0}\right)$, $C_t = \ln\left(H_t/S_{t,0}\right)$, and $X_t = \ln\left(S_{t,1}/S_{t,0}\right)$.

In practice, only four values of quotations during the day are usually available for each day t (the acquisition of intraday data is usually an added cost, and such data are not available for all assets): today's open price O_t, today's observed low price L_t^*, today's observed high price H_t^*, and today's closing price S_t. If today's open price O_t is different from yesterday's closing price S_{t-1} (the so-called night returns are nonzero), then the variables A_t and C_t can be redefined as: $A_t = \ln\left(\min\left(S_{t-1}, L_t^*\right)/S_{t-1}\right)$, $C_t = \ln\left(\max\left(S_{t-1}, H_t^*\right)/S_{t-1}\right)$ (see Fiszeder & Perczak, 2013).

For a standard Wiener process $\mathcal{B}_\tau$, $\tau \ge 0$, the Brownian motion $X_\tau^{\mathcal{B}} = \mu\tau + \sigma\mathcal{B}_\tau$ is defined. Let $s \in \mathbb{R}_+$ be a fixed value, $A_s^{\mathcal{B}} = \min_{0\le\tau\le s} X_\tau^{\mathcal{B}}$ and $C_s^{\mathcal{B}} = \max_{0\le\tau\le s} X_\tau^{\mathcal{B}}$. The probability density function of $X_s^{\mathcal{B}}$ with upper and lower absorbing barriers equal to c and a, respectively, is given by the formula (see Cox & Miller, 1965, p. 222, equation 78):

$$f_{X_s^{\mathcal{B}}}\left(a, c, x; \mu s, \sigma^2 s\right) = \frac{1}{dx} P\left(A_s^{\mathcal{B}} > a, C_s^{\mathcal{B}} \le c, X_s^{\mathcal{B}} \in dx\right)$$
$$= \frac{1}{\sqrt{2\pi}\sigma\sqrt{s}} e^{\frac{2\mu x - \mu^2 s}{2\sigma^2}} \times \sum_{k=-\infty}^{\infty}\left(e^{-\frac{(x-2k(c-a))^2}{2\sigma^2 s}} - e^{-\frac{(x-2c-2k(c-a))^2}{2\sigma^2 s}}\right), \quad (1)$$

where $a \le 0 \le c$, $a \le x \le c$.

The density of the random vector $\left(A_s^{\mathcal{B}}, C_s^{\mathcal{B}}, X_s^{\mathcal{B}}\right)$ can be expressed as (see Fiszeder & Perczak, 2013):

$$\begin{aligned}
& f_{A_s^{\mathcal{B}},C_s^{\mathcal{B}},X_s^{\mathcal{B}}}\left(a,c,x;\mu s,\sigma^2 s\right) \\
&= \frac{1}{dadcdx} P\left(A_s^{\mathcal{B}} \in da, C_s^{\mathcal{B}} \in dc, X_s^{\mathcal{B}} \in dx\right) \\
&= -\frac{\partial^2 f_{X_s^{\mathcal{B}}}\left(a,c,x;\mu s,\sigma^2 s\right)}{\partial a \partial c} \\
&= \frac{2\sqrt{2}e^{\frac{2\mu s x-\mu^2 s^2}{2\sigma^2 s}}}{\sqrt{\pi}\sigma^5 s^{5/2}} \sum_{k=-\infty}^{\infty} \left(g\left(a,c,x;k,k,\sigma^2 s\right)\right. \\
&\quad \left. - g\left(a,c,x;k,k+1,\sigma^2 s\right)\right), \qquad (2)
\end{aligned}$$

where the function g is described as:

$$\begin{aligned}
& g\left(a,c,x;k_1,k_2,\sigma^2 s\right) \\
&= k_1 k_2 \left[(x-2k_2 c+2k_1 a)^2 - \sigma^2 s\right] e^{-\frac{(x-2k_2 c+2k_1 a)^2}{2\sigma^2 s}}. \qquad (3)
\end{aligned}$$

When the density of the random vector $\left(A_s^{\mathcal{B}}, C_s^{\mathcal{B}}, X_s^{\mathcal{B}}\right)$ is defined by Eq. (2), it will have the distribution *ACN*, i.e., $\left(A_s^{\mathcal{B}}, C_s^{\mathcal{B}}, X_s^{\mathcal{B}}\right) \sim ACN\left(\mu s, \sigma^2 s\right)$.

A random variable W has the inverse Gaussian distribution denoted by $IG(\delta,\gamma)$, with parameters $0 < \delta$ and $0 \le \gamma$, if its probability density function $f_W(w,\delta,\gamma)$ determined for $w > 0$ has the form (see Barndorff-Nielsen, 1997, p. 2, equation 2.5):

$$f_W(w;\delta,\gamma) = \frac{\delta}{\sqrt{2\pi}} w^{-\frac{3}{2}} e^{\delta\gamma - \frac{\frac{\delta^2}{w}+\gamma^2 w}{2}}. \qquad (4)$$

If $\beta \in \mathbb{R}$, $\gamma \in \mathbb{R}_+$, $\delta \in \mathbb{R}_+$, $\mu \in \mathbb{R}$, $Z \sim N(0,1)$ and $W \sim IG(\delta,\gamma)$, then the random variable $X|W = \mu + \beta W + \sqrt{W}Z$ has the normal distribution $N(\mu+\beta W, W)$, while X has the NIG distribution $X \sim NIG(\alpha,\beta,\delta,\mu)$ for $\alpha = \sqrt{\beta^2+\gamma^2}$ (see Barndorff-Nielsen & Shephard, 2001, p. 15).

Let $s \in \mathbb{R}_+$ be a fixed parameter, let the random variable Y have the distribution $IG\left(\delta\sqrt{s}, \gamma\sqrt{s}\right)$, and let $\mathcal{B}_\tau$ be the Wiener process. If the process X_τ is described by the equation:

$$X_\tau | Y = \mu\tau + \beta Y \tau + \sqrt{Y}\mathcal{B}_\tau, \qquad (5)$$

then the following properties are met: $X_\tau|Y \sim N(\mu\tau + \beta Y\tau, Y\tau)$ and $Ys \sim IG(\delta s, \gamma)$. In addition, assuming that $W = Ys$, it is easy to show that we have $X_s \sim NIG(\alpha,\beta,\delta s,\mu s)$ at the point $\tau = s$.

The process X_τ is referred to as Brownian inverse Gaussian (*BIG*). Its specification is a modification of the process described by Venter, De Jongh, and Griebenow (2006, p. 102, equation 6).

Let $A_s = \min_{0\le\tau\le s} X_\tau$ and $C_s = \max_{0\le\tau\le s} X_\tau$; the density of X_s with upper and lower absorbing barriers equal to c and a, respectively, is given by the formula (see Perczak & Fiszeder, 2014; Venter et al., 2005, equation 4.5):

$$\begin{aligned}
& f_{X_s}(a,c,x;\alpha,\beta,\delta s,\mu s) \\
&= \frac{1}{dx} P(A_s > a, C_s \le c, X_s \in dx) \\
&= \int_0^\infty f_{X_s^{\mathcal{B}}|W=w}(a,c,x;\mu s+\beta w, w) f_W(w;\delta s,\gamma)\, dw \\
&= \alpha\kappa \sum_{k=-\infty}^{\infty} \left(\frac{K_1\left(\alpha\sqrt{\theta_1+\vartheta}\right)}{\sqrt{\theta_1+\vartheta}} - \frac{K_1\left(\alpha\sqrt{\theta_2+\vartheta}\right)}{\sqrt{\theta_2+\vartheta}}\right), \qquad (6)
\end{aligned}$$

where:
$\kappa = \frac{1}{\pi}\delta s e^{\beta(x-\mu s)+\delta\gamma s}$, $\theta_1 = (2k(c-a)-x)^2$, $\theta_2 = (2c+2k(c-a)-x)^2$, $\vartheta = \delta^2 s^2 + (x-\mu s)^2 - x^2$ and $K_\lambda(z) = \frac{1}{2}\int_0^\infty y^{\lambda-1} e^{-\frac{1}{2}z\left(y+\frac{1}{y}\right)} dy$.

Using Eq. (6), the joint density of the random vector (A_s, C_s, X_s) can be derived:

$$\begin{aligned}
& f_{A_s,C_s,X_s}(a,c,x;\alpha,\beta,\delta s,\mu s) \\
&= -\frac{\partial^2 f_{X_s}(a,c,x;\alpha,\beta,\delta s,\mu s)}{\partial a \partial c} \\
&= \int_0^\infty f_{A_s^{\mathcal{B}},C_s^{\mathcal{B}},X_s^{\mathcal{B}}|W=w}(a,c,x;\mu s+\beta w, w) \\
&\quad \times f_W(w;\delta s,\gamma)\, dw \\
&= 4\alpha^2\kappa \sum_{k=-\infty}^{\infty} \left[k^2 \Lambda(\alpha,\theta_1,\vartheta)\right. \\
&\quad \left. - k(k+1)\Lambda(\alpha,\theta_2,\vartheta)\right], \qquad (7)
\end{aligned}$$

where:

$$\begin{aligned}
\Lambda(\alpha,\theta,\vartheta) &= \alpha\theta \frac{K_1\left(\alpha\sqrt{\theta+\vartheta}\right)}{(\theta+\vartheta)^{\frac{3}{2}}} \\
&\quad + (3\theta-\vartheta)^2 \frac{K_2\left(\alpha\sqrt{\theta+\vartheta}\right)}{(\theta+\vartheta)^2}.
\end{aligned}$$

The random vector (A_s, C_s, X_s) will have the distribution denoted by *ACNIG*, i.e., $(A_s, C_s, X_s) \sim ACNIG(\alpha,\beta,\delta s,\mu s)$, when its density is defined by Eq. (7). Using relatively simple transformations of Eq. (7), this distribution can also be written in the different parameterization $ACNIG\left(\overline{\alpha},\overline{\beta},\delta s,\mu s\right)$, with invariant values $\overline{\alpha} = \alpha\delta s$, $\overline{\beta} = \beta\delta s$, $\overline{\gamma} = \sqrt{\overline{\alpha}^2 - \overline{\beta}^2}$:

$$\begin{aligned}
& f_{A_s,C_s,X_s}\left(a,c,x;\overline{\alpha},\overline{\beta},\delta s,\mu s\right) \\
&= 4\overline{\alpha}^2\overline{\kappa} \sum_{k=-\infty}^{\infty} \left[k^2 \Lambda\left(\frac{\overline{\alpha}}{\delta s},\theta_1,\vartheta\right)\right. \\
&\quad \left. - k(k+1)\Lambda\left(\frac{\overline{\alpha}}{\delta s},\theta_2,\vartheta\right)\right], \qquad (8)
\end{aligned}$$

where: $\overline{\kappa} = \frac{1}{\pi\delta s} e^{\overline{\beta}\frac{x-\mu s}{\delta s}+\overline{\gamma}}$.

Above, we showed that if X_τ is the process defined by Eq. (5), then $(A_s, C_s, X_s) \sim ACNIG\left(\overline{\alpha},\overline{\beta},\delta s,\mu s\right)$. This relationship does not have to be satisfied in the opposite direction. The random vector (A_s, C_s, X_s) may also have the $ACNIG\left(\overline{\alpha},\overline{\beta},\delta s,\mu s\right)$ distribution when X_τ is not the *BIG* process.

The next section assumes that, for a fixed value t $(t \in \mathbb{N},\ 0 < t)$, a return defined as $X_{t,\tau} = \ln\left(S_{t,\tau}/S_{t,0}\right)$ is the process for which $s = 1$. Thus, it follows that (A_t, C_t, X_t) has the *ACNIG* distribution. The exact parameters of this distribution are presented in the next section.

Based on the previous findings, the following implication holds:

$$W = sY \wedge Y \sim IG\left(\delta\sqrt{s}, \gamma\sqrt{s}\right) \wedge (A_s, C_s, X_s)\,|W \sim ACN\,(\mu s + \beta W, W)$$
$$\Downarrow$$
$$(A_s, C_s, X_s) \sim ACNIG\,(\alpha, \beta, \delta s, \mu s)\,. \tag{9}$$

The Rogers and Satchell (1991) estimator is an unbiased estimator of the variance of the arithmetic Brownian motion, i.e., $E\left[C_s\left(C_s - X_s\right) + A_s\left(A_s - X_s\right)|W\right] = W$. Based on the implication that $W \sim IG\,(\delta s, \gamma) \Rightarrow E\,[W] = \frac{\delta s}{\gamma}$, the following identities can be derived:

$$\begin{aligned} &E\left[C_s\left(C_s - X_s\right) + A_s\left(A_s - X_s\right)\right] \\ &\quad = E\left[E\left[C_s\left(C_s - X_s\right) + A_s\left(A_s - X_s\right)|W\right]\right] \\ &\quad = E\,[W] = \frac{\delta s}{\gamma}. \end{aligned} \tag{10}$$

The variance of a NIG-distributed random variable is well known. In particular, if $X_s \sim NIG\left(\overline{\alpha}, \overline{\beta}, \delta s, \mu s\right)$, then $Var\,[X_s] = \frac{\overline{\alpha}^2\delta^2 s^2}{\overline{\gamma}^3}$ (compare for example Barndorff-Nielsen, 1997, p. 2, formula 2.8; or Jensen & Lunde, 2001, p. 327).

It is desirable to find an unbiased estimator of the variance of the *BIG* process based on low, high and closing prices. It is easy to check that the expected value for the following estimator:

$$\begin{aligned} G\,(s) &= \frac{\alpha^2}{\gamma^2}\left[C_s\left(C_s - X_s\right) + A_s\left(A_s - X_s\right)\right] \\ &= \frac{\overline{\alpha}^2}{\overline{\gamma}^2}\left[C_s\left(C_s - X_s\right) + A_s\left(A_s - X_s\right)\right] \end{aligned} \tag{11}$$

is equal to:

$$E\,[G\,(s)] = \frac{\alpha^2}{\gamma^2}\frac{\delta s}{\gamma} = \frac{\alpha^2\delta s}{\gamma^3} = \frac{\overline{\alpha}^2\delta^2 s^2}{\overline{\gamma}^3}. \tag{12}$$

This implies that, for the known values of $\overline{\alpha}$ and $\overline{\beta}$, the equality holds $E\,[G\,(s)] = Var\,[X_s]$, which means that $G\,(s)$ is the required unbiased estimator of the variance of the *BIG* process.

3. The S&GARCH-NIG model for low, high and closing prices

Several extensions of the GARCH model based on low, high and closing prices were proposed by Perczak and Fiszeder (2014). We present the most complex model here (note that the S&GARCH-NIG-HLC and GARCH (1, 1) specifications are considered, but can be generalized to the GARCH (p, q) model):

$$(A_t, C_t, X_t)\,|\Im_{t-1} \sim ACNIG\left(\overline{\alpha}, \overline{\beta}, \frac{\overline{\gamma}^{\frac{3}{2}}}{\overline{\alpha}}\sqrt{h_t}, \mu\right), \tag{13}$$

$$\epsilon_t^2 = \frac{\overline{\alpha}^2}{\overline{\gamma}^2}\left[C_t\left(C_t - X_t\right) + A_t\left(A_t - X_t\right)\right], \tag{14}$$

$$h_t = \omega_0 + \omega_1\epsilon_{t-1}^2 + \xi_1 h_{t-1}. \tag{15}$$

This parameterization of the model has the following properties: $Var\,[\epsilon_t|\Im_{t-1}] = h_t$ and $X_t|\Im_{t-1} \sim NIG\left(\overline{\alpha}, \overline{\beta}, \frac{\overline{\gamma}^{\frac{3}{2}}}{\overline{\alpha}}\sqrt{h_t}, \mu\right)$. The parameters of Eq. (15) should ensure the positive variance and covariance stationarity of the process. In Eq. (14), we apply the new unbiased estimator of the daily variance presented in the previous section. This is a modification of the Rogers and Satchell (1991) estimator, and should be more efficient than the estimator calculated as the daily squared return of closing prices (as is commonly used in the standard GARCH model), due to the use of additional relevant information on the variability of prices during the day. We do not have a formal proof of this property at present, but it has been confirmed by Monte Carlo simulations.

The NIG conditional innovation distribution for the GARCH model is employed as a way of providing a better description of the fat tails of the distributions of most financial time series. This distribution was first used in the stochastic volatility model by Andersson (2001) and Barndorff-Nielsen (1997), before Jensen and Lunde (2001) applied it in their NIG-S&ARCH model. We prefer the NIG distribution to the Student-*t* distribution, which is the distribution that is used most commonly in empirical studies, because all of the moments of the distribution are finite. For example, the application of the Student-*t* distribution to logarithmic returns complicates the valuation of derivatives significantly (see Duan, 1999).

The model formulated in Eqs. (13)–(15) is a modification of the model proposed by Venter et al. (2005), applying the range estimator in Eq. (14) instead of the estimator calculated as the daily squared return of closing prices, and also assuming slight simplifications with different parameterizations of the random variables and stochastic processes that are employed.

It should be noted that the proposed model is parsimonious and there are no additional parameters relative to the model based only on the returns of closing prices.

The parameters of the model given in Eqs. (13)–(15) can be estimated by maximum likelihood with the likelihood function being based on low, high and closing prices:

$$\begin{aligned} &\left\{\widehat{\overline{\alpha}}, \widehat{\overline{\beta}}, \widehat{\omega}_0, \widehat{\omega}_1, \widehat{\xi}_1, \widehat{\mu}\right\} \\ &\quad = \underset{\{\overline{\alpha}, \overline{\beta}, \omega_0, \omega_1, \xi_1, \mu\}}{\arg\max}\ \ln L_{ACNIG}\left(\overline{\alpha}, \overline{\beta}, \omega_0, \omega_1, \xi_1, \mu\right) \\ &\quad = \underset{\{\overline{\alpha}, \overline{\beta}, \omega_0, \omega_1, \xi_1, \mu\}}{\arg\max}\ \sum_{t=1}^{n}\ln f_{ACNIG}\left(a_t, c_t, x_t, \overline{\alpha}, \overline{\beta},\right. \\ &\qquad \left.\frac{\overline{\gamma}^{\frac{3}{2}}}{\overline{\alpha}}\sqrt{h_t\left(\omega_0, \omega_1, \xi_1\right)}, \mu\right). \end{aligned} \tag{16}$$

4. Modelling the volatility of stock indices: S&P 500, FTSE 100 and WIG20

The usefulness of the model considered is illustrated by studying the three selected stock indices: S&P 500, FTSE

Table 1
Results of the estimations for the usual period.

Parameters	S&P 500			FTSE 100			WIG20		
	Model 1	Model 2	Model 3	Model 1	Model 2	Model 3	Model 1	Model 2	Model 3
$\mu \cdot 10^3$	0.465 (2.319)	3.002 (0.550)	0.391 (0.231)	0.213 (0.883)	5.265 (0.453)	0.538 (0.218)	3.325 (1.414)	8.002 (1.035)	1.154 (0.428)
$\omega_0 \cdot 10^6$	1.913 (1.184)	2.806 (1.104)	1.814 (1.111)	3.157 (1.651)	2.517 (0.802)	2.703 (1.111)	1.882 (1.313)	3.423 (1.269)	1.822 (1.266)
ω_1	0.042 (0.016)	0.108 (0.025)	0.043 (0.016)	0.079 (0.028)	0.114 (0.025)	0.081 (0.026)	0.038 (0.014)	0.097 (0.018)	0.036 (0.013)
ξ_1	0.913 (0.035)	0.816 (0.048)	0.915 (0.034)	0.846 (0.059)	0.824 (0.022)	0.854 (0.053)	0.950 (0.018)	0.875 (0.024)	0.953 (0.017)
$\overline{\alpha}$ or ν	8.464 (12.679)	2.548 (0.254)	24.994 (24.354)	3.981 (1.560)	2.576 (0.241)	15.652 (7.862)	2.640 (1.098)	3.492 (0.375)	7.934 (2.367)
$\overline{\beta}$ or ζ	−0.016 (1.047)	−0.775 (0.176)	0.942 (0.044)	−0.506 (0.315)	−1.358 (0.165)	0.847 (0.046)	−0.319 (0.243)	−1.210 (0.217)	0.959 (0.047)
$\ln L_1$	2729	2708	2729	2762	2735	2765	2270	2260	2268
RV	–	−2.541*	0.591	–	−2.846*	1.609	–	−1.496	−1.517
$\ln L_2$	10 493	10 763	–	10 771	10 873	–	9256	9313	–
RV	–	6.097*	–	–	3.602*	–	–	3.658*	–
LB(7)	7.614	7.511	7.621	10.241	11.239	9.759	13.132	11.548	13.247
LM(7)	2.794	4.076	2.804	12.108	17.952*	10.542	11.968	15.645*	12.941
AD	1.174	6.198*	1.062	0.355	3.760*	0.289	0.536	2.611*	0.408

Model 1 is the S&GARCH-NIG model for closing prices, Model 2 is the S&GARCH-NIG-HLC model, and Model 3 is the GARCH-skewed Student-*t* model. Standard errors are reported in parentheses. The parameters ν and ζ represent the degrees of freedom and the asymmetry of the skewed Student-*t* distribution, respectively. $\ln L_1$ and $\ln L_2$ are the logarithms of the likelihood functions based solely on closing prices, and based on low, high and closing prices, respectively. *RV* is the Rivers-Vuong test for model selection, where comparisons were made with the S&GARCH-NIG model, for which the parameters were estimated based only on closing prices as the benchmark. *LB*, *LM* and *AD* are the Ljung–Box test for the presence of autocorrelation, the Engle test for the presence of the ARCH effect and the Anderson–Darling goodness of fit test, respectively.
* Indicates that the null hypothesis was rejected at the 0.05 level.

100 and WIG20. The first two indices are those that are followed the most often by investors in equity indices representative of the USA and UK stock markets, while the WIG20 index is the Polish stock index quoted on the Warsaw Stock Exchange (after December 31, 2015, the WIG20 index will be replaced by a wider index, the WIG30). The Polish market is an example of a rapidly growing emerging market. There have been relatively few studies of Polish financial time series compared with other emerging or developed markets.

First, an initial valuation of the models considered was performed for daily logarithmic returns for the three-year period from January 2, 2004, to December 29, 2006 (757 returns). A shorter estimation period is not recommended, due to the relatively complicated likelihood functions required. Venter et al. (2005) introduced a similar model and used 1000 observations. In contrast, Forsberg and Bollerslev (2002) and Jensen and Lunde (2001) applied the GARCH-NIG and NIG-S&ARCH models and used almost 10 years and 30 years of daily returns, respectively.

The reference model is the S&GARCH-NIG, where the parameters are estimated based only on closing prices, i.e., an extension of the model of Jensen and Lunde (2001) to the GARCH specification. The second model is the proposed S&GARCH-NIG-HLC model given in Eqs. (13)–(15), where the parameters are estimated based on low, high and closing prices. As was mentioned in the introduction, this parameterization is a slight modification of the model proposed by Venter et al. (2005). A natural competitor for the NIG distribution is the skewed Student-*t* distribution. Both distributions can describe such asymmetry and leptokurtosis, which is why the comparison also includes the GARCH model with the skewed Student-*t* conditional innovation distribution (which we refer to as GARCH-skewed Student-*t*), where the parameters are estimated based on closing prices (for a detailed specification of the model, see e.g. Osiewalski & Pipień, 1999). The Student-*t* distribution is the distribution that is used most frequently in empirical applications for financial series. There was no statistically significant autocorrelation among the returns, which is why there is only a constant in the conditional mean equation. The parameters of these models are estimated using the maximum likelihood method. All three models have the same numbers of parameters (the other extensions of the GARCH model based on low, high and closing prices presented by Perczak and Fiszeder (2014), are omitted deliberately because they are special cases of the S&GARCH-NIG-HLC model). The results of the estimations are presented in Table 1.

The logarithms of the likelihood function for the S&GARCH-NIG and GARCH-skewed Student-*t* models are based solely on closing prices (indicated by $\ln L_1$). For purposes of information only, the logarithms of the likelihood function $\ln L_{ACNIG}$ based on low, high and closing prices were also calculated (denoted $\ln L_2$). During the estimation of the parameters of the S&GARCH-NIG-HLC model, the value of $\ln L_2$ was maximized, but the value of $\ln L_1$ was also calculated for information purposes. The measure $\ln L_2$ includes more relevant information on the variability in the prices of financial instruments (low and high prices certainly comprise valid information from the point of view of volatility measurement, see e.g. Chou et al., 2010), which is why we can assume that it is more reliable

Table 2
Results estimated for the turbulent period during the financial crisis in the USA.

Parameters	S&P 500			FTSE 100			WIG20		
	Model 1	Model 2	Model 3	Model 1	Model 2	Model 3	Model 1	Model 2	Model 3
$\mu \cdot 10^3$	2.159 (0.631)	3.745 (0.543)	0.255 (0.413)	1.276 (0.896)	4.114 (0.707)	0.367 (0.411)	−1.668 (2.282)	4.473 (1.532)	−0.120 (0.650)
$\omega_0 \cdot 10^6$	1.881 (1.437)	1.435 (0.775)	1.617 (1.328)	3.081 (1.647)	2.154 (1.103)	2.707 (1.512)	5.378 (3.326)	7.164 (2.363)	5.316 (3.299)
ω_1	0.117 (0.025)	0.289 (0.029)	0.123 (0.024)	0.122 (0.026)	0.252 (0.032)	0.120 (0.024)	0.061 (0.017)	0.146 (0.020)	0.059 (0.017)
ξ_1	0.888 (0.021)	0.754 (0.023)	0.885 (0.019)	0.872 (0.024)	0.756 (0.030)	0.875 (0.022)	0.927 (0.020)	0.832 (0.022)	0.928 (0.020)
$\overline{\alpha}$ or ν	1.089 (0.305)	1.847 (0.171)	5.522 (1.386)	2.394 (0.864)	2.536 (0.248)	9.117 (3.058)	5.085 (2.671)	2.677 (0.271)	12.977 (5.593)
$\overline{\beta}$ or ζ	−0.159 (0.070)	−0.533 (0.094)	0.858 (0.040)	−0.110 (0.122)	−0.622 (0.122)	0.909 (0.047)	0.204 (0.300)	−0.522 (0.178)	1.000 (0.054)
$\ln L_1$	2153	2141	2154	2186	2188	2186	1912	1888	1912
RV	–	−1.108	0.231	–	0.265	0.570	–	−2.444*	−0.364
$\ln L_2$	9126	9242	–	9022	9105	–	8110	8339	–
RV	–	4.712*	–	–	3.224*	–	–	5.435*	–
LB(7)	11.988	13.898	12.396	13.571	14.417*	13.592	5.867	6.104	5.872
LM(7)	4.951	2.162	5.534	7.561	10.806	7.494	18.905*	8.573	18.729*
AD	0.734	8.877*	1.329	0.588	2.991*	0.951	0.361	9.565*	0.291

Model 1 is the S&GARCH-NIG model for closing prices, Model 2 is the S&GARCH-NIG-HLC model, and Model 3 is the GARCH-skewed Student-*t* model. Standard errors are reported in parentheses. The parameters ν and ζ represent the degrees of freedom and the asymmetry of the skewed Student-*t* distribution, respectively. $\ln L_1$ and $\ln L_2$ are the logarithms of the likelihood function based solely on closing prices, and based on low, high and closing prices, respectively. *RV* is the Rivers-Vuong test for model selection, where comparisons were made with the S&GARCH-NIG model, for which the parameters were estimated based only on closing prices as the benchmark. *LB*, *LM* and *AD* are the Ljung–Box test for the presence of autocorrelation, the Engle test for the presence of the ARCH effect and the Anderson–Darling goodness of fit test, respectively.
* Indicates that the null hypothesis was rejected at the 0.05 level.

as a measure of the model's quality. We also performed the Rivers and Vuong (2002) (RV) test, which allowed us to verify the hypothesis that the likelihood functions of two non-nested competing models are asymptotically equivalent. The RV test is a generalization of the Vuong tests (1989), which can be applied to nonlinear models of time series. According to the likelihood function based solely on closing prices, the S&GARCH-NIG-HLC model was inferior to the models based solely on closing prices for the S&P 500 and FTSE 100 indices, but there were no significant differences between the competing models for the WIG20 index. However, when the likelihood function was based on low, high and closing prices ($\ln L_2$), the S&GARCH-NIG-HLC model performed significantly better than the S&GARCH-NIG model for all of the indices (it was not possible to compare the GARCH-skewed Student-*t* model directly).

The evaluation of the quality of the models was based on statistical tests for closing prices: the Ljung–Box test for the presence of autocorrelation, the Engle test for the presence of the ARCH effect, and the Anderson–Darling test for the goodness of fit. The application of low and high prices to the estimation of parameters degraded the quality of the GARCH model evaluations from the point of view of the statistical properties of the errors formulated for closing prices (a weak ARCH effect for the FTSE 100 and WIG20 indices, and a worse fit of the conditional distribution for the three indices). This could be expected, and does not mean that the model was of low quality. Thus, the models should be evaluated based on a broader set of information related to low, high and closing prices, but no appropriate procedures or tests are available at present (we leave this for future research).

The application of low and high prices to estimation changed the estimates of the GARCH model parameters significantly. Specifically, the estimate of the parameter ω_1 increased and the estimate of the parameter ξ_1 decreased compared with the models where the parameters were estimated based on closing prices. This is important in terms of both the modelling and forecasting of the volatility of returns, because shocks in the previous period have a stronger impact on the current volatility, and thus, the model using parameters estimated based on low, high and closing prices has a faster response to changing market conditions. One of the greatest weaknesses of the GARCH model using parameters estimated based on closing prices is cited widely as being its slow response to abrupt changes in the market (e.g., Andersen et al., 2003; Hansen et al., 2012). Thus, it seems that the use of low and high prices for estimating parameters yields estimates that are closer to the true parameters.

It is well known that the extreme values connected with turbulent periods have significant effects on the estimation results. Therefore, it is of interest to compare the estimated parameters of the GARCH models during the tranquil period with the results obtained during the financial crisis in the USA. The latter estimates were obtained for a turbulent period of the same length as the tranquil period, namely from January 2, 2007, to December 31, 2009. The estimated results are presented in Table 2.

The estimate of the parameter ω_1 increased during the turbulent period relative to the tranquil period before the

Table 3
Summary statistics of the daily returns for tranquil and turbulent periods.

Index	Number of observations	Mean · 10^3	Minimum	Maximum	Standard deviation	Skewness	Excess kurtosis
The tranquil period: year 2007							
S&P 500	251	0.138	−0.035	0.029	0.010	−0.494*	1.448*
FTSE 100	253	0.147	−0.042	0.034	0.011	−0.366*	1.537*
WIG20	249	0.203	−0.055	0.045	0.015	0.000	0.636*
The turbulent period: year 2008							
S&P 500	253	−1.921	−0.095	0.110	0.026	−0.033	3.672*
FTSE 100	254	−1.480	−0.093	0.094	0.024	0.126	3.383*
WIG20	251	−2.622	−0.084	0.081	0.024	−0.210	1.445*
The transition period: year 2009							
Athex Composite Share Price	248	0.832	−0.064	0.069	0.021	−0.202	0.648*
The turbulent period: year 2011							
Athex Composite Share Price	251	−2.914	−0.072	0.134	0.023	0.969*	4.773*

* Indicates that the null hypothesis (the skewness or excess kurtosis is equal to zero) was rejected at the 0.05 level.

crisis for all of the models. The estimate of the parameter ξ_1 decreased and the sum of the estimates of parameters $\omega_1 + \xi_1$ increased (as expected, it was very close to one) during the financial crisis in almost all cases. The greatest changes were for the S&GARCH-NIG-HLC model. The parameters estimated for the two additional periods of 2005–2007 and 2006–2008 are given in Tables A.1 and A.2 in the Appendix, which show how the estimates of the parameters reacted to the period of turmoil.

5. Forecasting volatility

The main purpose of this empirical study is to compare the forecasting performance of the GARCH model estimated based on low, high and closing prices with that of the model based only on closing prices in the turbulent period, i.e., the period with large declines in stock prices and very high volatilities. The forecasting performances of the models were evaluated in the period of turmoil, i.e., for the whole year of 2008, which was when the worst phase of the financial crisis occurred in the USA and the turbulence spread to the European financial markets. The stock indices analysed, i.e., S&P 500, FTSE 100 and WIG20, lost 38.49%, 31.33% and 48.21% of their values in 2008, respectively. The choice of this period was subjective, but results similar to those presented in the remainder of this study were obtained when slightly different turbulent periods were assumed. For comparison, the forecasts were also evaluated for the preceding usual period of a similar length (in this case, 2007). In 2007, the S&P 500, FTSE 100 and WIG20 indices increased by 3.53%, 3.80% and 5.19%, respectively. It should be noted that the choice of period is not important because no significant conclusions are drawn for this period.

The descriptive statistics for the daily logarithmic returns presented in Table 3 confirm the significant differences in returns between 2007 and 2008. The variability of returns, measured by the standard deviation, increased in the year 2008 by about 156%, 114% and 61% for the S&P 500, FTSE 100 and WIG20 indices, respectively. The results for the Athex Composite Share Price Index, which is analyzed in Section 6, are also given in Table 3.

Out-of sample one-day-ahead forecasts of the variance were formulated based on the models, where the parameters were estimated separately each day based on a rolling sample with a fixed size of 757 (approximately a three-year period) for the years 2007 and 2008. As a proxy of the daily volatility for the evaluation of forecasts, we employed the sum of the squared intraday returns (the so-called realized variance). One significant problem when using such data is the choice of the appropriate frequency of observations (see, e.g. Pigorsch, Pigorsch, & Popov, 2012). Therefore, the realized volatility was estimated in different variants using 5, 10, 15, 20, 30 and 60 min returns (to save space, only the results for 15 min returns are presented below; however, the other frequencies were not significantly different). The forecasts of the models were evaluated based on two primary measures, namely the mean squared error (MSE) and the mean absolute error (MAE). The MSE is the criterion that is used most frequently in empirical studies, and is also robust to the use of a noisy volatility proxy (it yields the same ranking of competing forecasts using an unbiased volatility proxy, see Hansen & Lunde, 2006a; Patton, 2011). However, the MAE is less sensitive to outliers. The results of the study when the ex-post realized variances were estimated as the sum of the squared 15-min returns are presented in Table 4.

In order to evaluate the statistical significance of the results, two different tests were applied, the test of superior predictive ability (SPA) of Hansen (2005), and the model confidence set (MCS) of Hansen, Lunde, and Nason (2011). In the first approach, alternative forecasts are compared with a benchmark forecast. In this study, we performed a pairwise comparison, and the results are presented with respect to the S&GARCH-NIG model based on closing prices used as a benchmark. In contrast, the MCS procedure does not require the specification of a benchmark model. The MCS contains the best forecasting models with a certain probability.

According to the results of the SPA tests for the MSE and MAE criteria (see the outcomes in Table 4, noting

Table 4
Evaluation of the volatility forecasts: the MSE and MAE criteria. The realized variance is used as a proxy for the volatility.

Model	The tranquil period: year 2007						The turbulent period: year 2008					
	MSE	SPA *p*-value	MCS *p*-value	MAE	SPA *p*-value	MCS *p*-value	MSE	SPA *p*-value	MCS *p*-value	MAE	SPA *p*-value	MCS *p*-value
S&P 500												
Model 1	0.083	–	0.262*	0.532	–	0.164*	5.628	–	0.002	4.143	–	0.000
Model 2	0.092	0.859	0.218*	0.496	0.083	1.000*	3.303	0.003	1.000*	2.610	0.000	1.000*
Model 3	0.082	0.141	1.000*	0.537	0.961	0.163*	5.798	0.894	0.002	4.204	0.929	0.000
FTSE 100												
Model 1	0.118	–	0.851*	0.574	–	0.310*	5.754	–	0.094	3.340	–	0.044
Model 2	0.119	0.567	0.851*	0.546	0.143	1.000*	4.777	0.040	1.000*	2.824	0.026	1.000*
Model 3	0.117	0.077	1.000*	0.573	0.213	0.310*	5.590	0.015	0.098	3.288	0.005	0.051
WIG20												
Model 1	0.427	–	1.000*	1.093	–	0.929*	6.289	–	0.024	3.792	–	0.004
Model 2	0.465	0.917	0.101*	1.086	0.418	1.000*	5.290	0.029	1.000*	3.289	0.004	1.000*
Model 3	0.429	0.718	1.666*	1.089	0.105	0.929*	6.221	0.085	0.024	3.742	0.005	0.004

Model 1 is the S&GARCH-NIG model for closing prices, Model 2 is the S&GARCH-NIG-HLC model, and Model 3 is the GARCH-skewed Student-*t* model. The realized variances were estimated as the sum of squared 15-min returns. The values of the MSE and MAE are multiplied by 10^7 and 10^4, respectively. The SPA test is performed for pairs of models, with the S&GARCH-NIG using closing prices as a benchmark. The MCS test is performed for the three models jointly.

* Indicates that models belong to the MCS with a confidence level of 0.90.

that the results were the same when other models were used as benchmarks, though they are not presented here in order to save space), there were no differences (at the 5% significance level) between the forecasts based on the three GARCH models during the usual period. Similarly, according to the MCS test for both the MSE and MAE criteria, all three models belonged to the MCS and there was no evidence to reject the null hypothesis of equal predictive ability.

However, completely different results were obtained for the turbulent period. The results of the SPA tests for the MSE and MAE loss functions (see the outcomes in Table 4, noting that the results were the same when the GARCH-skewed Student-*t* model was used as the benchmark, and therefore they are not presented here in order to save space) indicated that the forecasts from the models where the parameters were estimated based only on closing prices were inferior to the forecasts from the models that used parameters based on low, high and closing prices. The same conclusion was obtained based on the results of the MCS tests for both the MSE and MAE measures. Only the S&GARCH-NIG-HLC model belonged to the MSC with 0.90 confidence, thus indicating that it is the best forecasting model. The main conclusions of this study did not depend on the loss function employed, and the results were the same for both the MSE and MAE measures.

It should be noted that the forecasting errors were significantly lower for the S&P 500 and FTSE 100 indices than for the WIG20 index during the tranquil period, namely the year 2007, while the sizes of the errors were more similar during the turbulent period.

Other loss functions were also considered, but yielded similar results. Thus, to save space, Table 5 only presents the R^2 values from the Mincer–Zarnowitz regression.

The MSE and MAE measures indicated a significant degradation in the quality of the forecasts during the period of turmoil (at least in terms of their absolute values).

Table 5
Evaluation of the volatility forecasts: coefficient of determination.

Model	Indices		
	S&P 500	FTSE 100	WIG20
The tranquil period: year 2007			
Model 1	0.243	0.176	0.091
Model 2	0.314	0.140	0.049
Model 3	0.252	0.183	0.089
The turbulent period: year 2008			
Model 1	0.417	0.238	0.296
Model 2	0.539	0.328	0.407
Model 3	0.439	0.252	0.309

Model 1 is the S&GARCH-NIG model for closing prices, Model 2 is the S&GARCH-NIG-HLC model, and Model 3 is the GARCH-skewed Student-*t* model. The sum of the squared 15-min returns is used as a proxy for the volatility.

Note, though, that the R^2 values suggest a completely different conclusion; however, the latter is a relative measure and does not penalize biased forecasts. It should be mentioned that the volatility forecasts based on the GARCH model with parameters estimated based on low, high and closing prices showed a huge increase in accuracy in the turbulent period compared with the usual period for all three indices.

6. Robustness check for additional proxies of volatility and a different period of turmoil

Two different additional proxies for the daily volatility were employed in order to check the robustness of the forecasting results: the first-order autocorrelation-adjusted realized variance estimator and the bi-power variation. The former is the realized variance plus twice the sum of the products of adjacent intraday returns. This measure is designed to capture the effect of autocorrelation

Table 6
Evaluation of the volatility forecasts using other volatility proxies: the MSE and MAE criteria for the tranquil period (2007).

Model	First-order autocorrelation-adjusted RV						Bi-power variation					
	MSE	SPA *p*-value	MCS *p*-value	MAE	SPA *p*-value	MCS *p*-value	MSE	SPA *p*-value	MCS *p*-value	MAE	SPA *p*-value	MCS *p*-value
S&P 500												
Model 1	0.078	–	0.441*	0.546	–	0.154*	0.045	–	0.230*	0.475	–	0.000
Model 2	0.087	0.875	0.249*	0.507	0.074	1.000*	0.038	0.110	1.000*	0.363	0.000	1.000*
Model 3	0.077	0.212	1.000*	0.549	0.882	0.154*	0.046	0.911	0.230*	0.485	0.502	0.000
FTSE 100												
Model 1	0.170	–	0.654*	0.649	–	0.540*	0.121	–	0.807*	0.571	–	0.335*
Model 2	0.175	0.661	0.654*	0.632	0.241	1.000*	0.123	0.591	0.807*	0.545	0.157	1.000*
Model 3	0.168	0.059	1.000*	0.647	0.172	0.540*	0.120	0.083	1.000*	0.571	0.354	0.335*
WIG20												
Model 1	0.422	–	1.000*	1.202	–	0.594*	0.401	–	1.000*	1.131	–	0.086
Model 2	0.471	0.944	0.048	1.206	0.551	0.594*	0.423	0.841	0.236*	1.066	0.044	1.000*
Model 3	0.425	0.752	0.586*	1.195	0.054	1.000*	0.401	0.572	0.939*	1.126	0.076	0.086

Model 1 is the S&GARCH-NIG model for closing prices, Model 2 is the S&GARCH-NIG-HLC model, and Model 3 is the GARCH-skewed Student-*t* model. The other volatility proxies are employed for 15-min returns. The values of the MSE and MAE are multiplied by 10^7 and 10^4, respectively. The SPA test is performed for pairs of models, with the S&GARCH-NIG using closing prices as the benchmark. The MCS test is performed for all three models jointly.

* Indicates that models belong to the MCS with a confidence level of 0.90.

Table 7
Evaluation of volatility forecasts for other volatility proxies: the MSE and MAE criteria for the turbulent period (2008).

Model	First-order autocorrelation-adjusted RV						Bi-power variation					
	MSE	SPA *p*-value	MCS *p*-value	MAE	SPA *p*-value	MCS *p*-value	MSE	SPA *p*-value	MCS *p*-value	MAE	SPA *p*-value	MCS *p*-value
S&P 500												
Model 1	6.657	–	0.005	4.428	–	0.000	5.618	–	0.000	4.175	–	0.000
Model 2	4.439	0.007	1.000*	2.952	0.000	1.000*	2.501	0.002	1.000*	2.302	0.000	1.000*
Model 3	6.845	0.893	0.006	4.504	0.505	0.000	6.002	0.526	0.000	4.319	0.525	0.000
FTSE 100												
Model 1	7.248	–	0.228*	3.990	–	0.041	3.362	–	0.121*	3.008	–	0.051
Model 2	6.432	0.080	1.000*	3.498	0.019	1.000*	2.570	0.049	1.000*	2.500	0.033	1.000*
Model 3	7.064	0.017	0.257*	3.931	0.002	0.049	3.242	0.007	0.135*	2.963	0.008	0.054
WIG20												
Model 1	7.057	–	0.049	4.043	–	0.020	6.077	–	0.028	3.740	–	0.003
Model 2	6.204	0.041	1.000*	3.611	0.010	1.000*	5.166	0.031	1.000*	3.237	0.003	1.000*
Model 3	6.976	0.085	0.049	4.002	0.016	0.021	6.014	0.087	0.028	3.688	0.004	0.004

Model 1 is the S&GARCH-NIG model for closing prices, Model 2 is the S&GARCH-NIG-HLC model, and Model 3 is the GARCH-skewed Student-*t* model. The other volatility proxies are employed for 15-min returns. The values of the MSE and MAE are multiplied by 10^7 and 10^4, respectively. The SPA test is performed for pairs of models, with the S&GARCH-NIG using closing prices as the benchmark. The MCS test is performed for the three models jointly.

* Indicated that models belong to the MCS with a confidence level of 0.90.

in high frequency returns induced by market microstructure noise (such as the bid–ask bounce). This estimator was studied extensively by Hansen and Lunde (2006b). The bi-power variation of Barndorff-Nielsen and Shephard (2004) is the scaled sum of the products of adjacent absolute intraday returns. This is a jump-robust realized measure. The forecasting performance results obtained by the GARCH models using these two proxies of the daily volatility are presented in Tables 6 and 7, respectively, for the usual and turbulent periods. The differences are small compared with the results reported in Section 5. The S&GARCH-NIG-HLC model performed better than its competitors during the usual period for the S&P 500 and WIG20 indices when employing the MAE criterion and the bi-power variation proxy for volatility. During the turbulent period, all three models belonged to the MSC with 0.90 confidence for the FTSE 100 index based on the MSE measure. However, the other results agreed with the main conclusions of the study.

The results from this research may apply only to the financial crisis in the USA. Thus, to check the robustness of the forecasting results, a different period of turmoil was considered as well. This study was performed for the Greek debt crisis using the Athex Composite Share Price Index. First, the parameters of all three models were estimated for the turbulent period of 2010–2012. It was not possible to estimate the parameters for the tranquil period before the crisis, due to the overlap with the previous three-year period of the financial crisis in the USA; this is why the mixed period of 2009–2011 was also considered for the comparison. The results are presented in Table 8. The main outcomes are similar to those obtained for the USA crisis.

Table 8
Results estimated for the Athex Composite Share Price index during the Greek debt crisis.

Parameters	The usual and turbulent periods, 2009–2011			The turbulent period, 2010–2012		
	Model 1	Model 2	Model 3	Model 1	Model 2	Model 3
$\mu \cdot 10^3$	−2.967 (2.441)	4.480 (1.872)	−0.819 (0.771)	−3.970 (2.239)	−0.540 (1.790)	−0.906 (0.799)
$\omega_0 \cdot 10^6$	17.097 (10.156)	20.063 (5.966)	16.180 (9.355)	30.194 (13.465)	21.363 (5.497)	29.339 (12.900)
ω_1	0.044 (0.017)	0.140 (0.024)	0.046 (0.017)	0.079 (0.024)	0.181 (0.025)	0.084 (0.024)
ξ_1	0.919 (0.031)	0.804 (0.034)	0.920 (0.029)	0.866 (0.037)	0.771 (0.031)	0.863 (0.036)
$\overline{\alpha}$ or ν	2.655 (0.932)	2.726 (0.280)	8.654 (2.471)	2.304 (0.744)	2.665 (0.274)	7.751 (1.967)
$\overline{\beta}$ or ζ	0.159 (0.200)	−0.553 (0.196)	1.072 (0.055)	0.207 (0.170)	−0.056 (0.172)	1.090 (0.057)
$\ln L_1$	1829	1808	1830	1766	1753	1767
RV	–	−2.530*	0.854	–	−1.516	1.025
$\ln L_2$	7997	8134	–	7783	7872	–
RV	–	4.408*	–	–	3.492*	–
LB(7)	7.782	7.111	7.745	4.798	4.986	4.703
LM(7)	8.308	9.527	8.651	2.901	3.750	3.245
AD	0.381	8.680*	0.382	0.294	6.921*	0.283

Model 1 is the S&GARCH-NIG model for closing prices, Model 2 is the S&GARCH-NIG-HLC model, and Model 3 is the GARCH-skewed Student-*t* model. Standard errors are reported in parentheses. The parameters ν and ζ represent the degrees of freedom and the asymmetry of the skewed Student-*t* distribution, respectively. $\ln L_1$ and $\ln L_2$ are the logarithms of the likelihood function based solely on closing prices, and based on low, high and closing prices, respectively. *RV* is the Rivers-Vuong test for model selection, where comparisons are made using the S&GARCH-NIG model with the parameters estimated based only on closing prices as the benchmark. *LB*, *LM* and *AD* are the Ljung–Box test for the presence of autocorrelation, the Engle test for the presence of the ARCH effect and the Anderson–Darling goodness of fit test, respectively.
* Indicates that the null hypothesis was rejected at the 0.05 level.

Table 9
Evaluation of the volatility forecasts for different volatility proxies and the MSE and MAE criteria for the Athex Composite Share Price index during the transition and turbulent periods of the Greek debt crisis.

Model	The transition period: year 2009						The turbulent period: year 2011					
	MSE	SPA *p*-value	MCS *p*-value	MAE	SPA *p*-value	MCS *p*-value	MSE	SPA *p*-value	MCS *p*-value	MAE	SPA *p*-value	MCS *p*-value
Realized variance												
Model 1	1.142	–	0.102*	2.453	–	0.014	3.532	–	0.029	3.237	–	0.000
Model 2	0.890	0.058	1.000*	2.058	0.011	1.000*	3.235	0.027	1.000*	2.477	0.000	1.000*
Model 3	1.111	0.021	0.112*	2.390	0.000	0.023	3.542	0.880	0.029	3.247	0.941	0.000
First-order autocorrelation-adjusted RV												
Model 1	1.461	–	0.186*	2.740	–	0.013	4.777	–	0.113*	3.510	–	0.000
Model 2	1.250	0.096	1.000*	2.353	0.011	1.000*	4.564	0.065	1.000*	2.982	0.000	1.000*
Model 3	1.444	0.154	0.186*	2.681	0.001	0.020	4.784	0.794	0.113*	3.516	0.860	0.000
Bi-power variation												
Model 1	1.182	–	0.034	2.569	–	0.005	1.507	–	0.000	2.839	–	0.000
Model 2	0.817	0.022	1.000*	2.085	0.003	1.000*	0.832	0.002	1.000*	1.809	0.000	1.000*
Model 3	1.128	0.003	0.043	2.475	0.000	0.010	1.531	0.054	0.000	2.853	0.498	0.000

Model 1 is the S&GARCH-NIG model for closing prices, Model 2 is the S&GARCH-NIG-HLC model, and Model 3 is the GARCH-skewed Student-*t* model. Volatility proxies are employed for 15-min returns. The values of the MSE and MAE are multiplied by 10^7 and 10^4, respectively. The SPA test is performed for pairs of models, with the S&GARCH-NIG using closing prices as the benchmark. The MCS test is performed for the three models jointly.
* Indicates that models belong to the MCS with a confidence level of 0.90.

The analysis of the forecasting performance was performed for the year 2011, i.e., at the worst stage of the crisis, and for the year 2009, for comparison. The year 2010 was omitted deliberately because it was the beginning of the debt crisis (the results for the year 2010 are presented in Table A.3 in the Appendix). The main stock index lost more than half of its value (51.88%) in 2011, after increasing by 22.93% in 2009. The descriptive statistics of the index returns are presented in Table 3, and the forecasting performance results are given in Table 9. The year 2009 was not a tranquil period, but the transition period between the financial crisis in the USA and the debt crisis in Greece. Despite the growth in the market, the volatility of returns was very high in 2009 (only slightly lower than that in 2011), and the forecasting superiority of the S&GARCH-NIG-HLC model was also noticeable in that period.

The R^2 values obtained from the Mincer–Zarnowitz regression were 0.087, 0.082 and 0.082 in the year 2009, and 0.051, 0.091 and 0.052 in the year 2011, for the S&GARCH-NIG based on closing prices, S&GARCH-NIG-HLC and GARCH-skewed Student-*t* models, respectively.

Thus, the main conclusion of the study remains unchanged, and the results indicate that, during the turbulent period, the forecasts obtained using the models with parameters estimated based only on closing prices were inferior to those produced by the models where the parameters were estimated based on low, high and closing prices.

7. Conclusions

Forecasting the volatility of financial asset returns is a much easier task than forecasting financial asset returns. Nevertheless, forecasting the variance of returns during the turbulent periods connected with financial crises is a difficult task, and traditional volatility models do not cope well with this problem. The GARCH model, which is used widely in empirical studies, is not well suited to situations with rapid changes in the level of volatility. This is due in part to the fact that its formulation and the estimation of its parameters are based solely on daily closing prices, which do not always provide satisfactory information about the volatility of returns. By contrast, much more information can be obtained through the use of daily minimum and maximum prices, which are available widely.

In this study, we have proposed a modification of the GARCH model whereby the parameters are estimated based on the daily low and high prices as well as the closing prices. We have also presented an empirical application to two stock indices from developed markets, i.e., S&P 500 and FTSE 100, and two stock indices from emerging markets, i.e., the Polish WIG20 index and the Greek Athex Composite Share Price Index. We used the additional information on low and high prices in the derivation of the likelihood function of the GARCH model, which improved the volatility estimation and increased the accuracy of the volatility forecasts for the models in the turbulent period compared with using only closing prices. This result was robust to both the forecast evaluation criterion employed and the proxy used for the daily volatility. In future, this method could be extended to other GARCH models, as well as to other volatility models such as the stochastic volatility model.

Acknowledgments

This research was financed by the National Science Centre, project number 2012/05/B/HS4/00675, entitled "Modelling and Forecasting Volatility—Usage of Additional Information Contained in Low and High Prices". The authors would like to thank two anonymous reviewers for their constructive comments and useful recommendations.

Appendix

See Tables A.1–A.3.

Table A.1
Estimates obtained for selected stock indices over the period 2005–2007.

Parameters	S&P 500			FTSE 100			WIG20		
	Model 1	Model 2	Model 3	Model 1	Model 2	Model 3	Model 1	Model 2	Model 3
$\mu \cdot 10^3$	1.541 (0.544)	3.262 (0.453)	0.404 (0.242)	1.642 (0.796)	4.770 (0.426)	0.623 (0.231)	5.650 (2.425)	9.421 (1.057)	1.077 (0.487)
$\omega_0 \cdot 10^6$	1.334 (0.740)	1.592 (0.575)	1.167 (0.690)	2.916 (1.171)	1.943 (0.621)	2.512 (1.031)	4.618 (2.693)	5.901 (2.395)	4.487 (2.792)
ω_1	0.067 (0.019)	0.133 (0.022)	0.069 (0.018)	0.138 (0.033)	0.188 (0.028)	0.129 (0.030)	0.053 (0.018)	0.106 (0.020)	0.044 (0.017)
ξ_1	0.912 (0.026)	0.840 (0.027)	0.913 (0.025)	0.815 (0.044)	0.785 (0.032)	0.830 (0.040)	0.924 (0.026)	0.857 (0.029)	0.934 (0.026)
$\overline{\alpha}$ or ν	1.703 (0.525)	2.095 (0.197)	6.791 (1.681)	4.798 (2.317)	2.208 (0.199)	22.575 (16.800)	5.276 (2.647)	2.805 (0.276)	11.218 (4.530)
$\overline{\beta}$ or ζ	−0.212 (0.124)	−0.734 (0.131)	0.892 (0.042)	−0.324 (0.259)	−1.054 (0.127)	0.832 (0.045)	−0.822 (0.573)	−1.221 (0.181)	0.962 (0.051)
$\ln L_1$	2661	2641	2662	2658	2620	2663	2172	2153	2169
RV	–	−2.426*	0.795	–	−3.161*	1.799	–	−2.685*	−1.136
$\ln L_2$	10 539	10 612	–	10 422	10 570	–	8914	9048	–
RV	–	3.507*	–	–	4.243*	–	–	4.894*	–
LB(7)	10.865	11.416	10.636	7.526	2.731	6.829	8.532	8.140	7.962
LM(7)	4.468	4.442	4.728	5.188	27.091*	4.658	11.689	10.238	11.462
AD	0.564	5.549*	0.748	0.652	4.919*	0.640	0.574	5.499*	0.553

Model 1 is the S&GARCH-NIG model for closing prices, Model 2 is the S&GARCH-NIG-HLC model, and Model 3 is the GARCH-skewed Student-*t* model. Standard errors are reported in parentheses. The parameters ν and ζ represent the degrees of freedom and the asymmetry of the skewed Student-*t* distribution, respectively. $\ln L_1$ and $\ln L_2$ are the logarithms of the likelihood function based solely on closing prices, and based on low, high and closing prices, respectively. *RV* is the Rivers-Vuong test for model selection, where comparisons are made using the S&GARCH-NIG model with parameters estimated based only on closing prices as the benchmark. *LB*, *LM* and *AD* are the Ljung–Box test for the presence of autocorrelation, the Engle test for the presence of the ARCH effect and the Anderson–Darling goodness of fit test, respectively.

* Indicates that the null hypothesis was rejected at the 0.05 level.

Table A.2
Results estimated for selected stock indices over the period 2006–2008.

Parameters	S&P 500			FTSE 100			WIG 20		
	Model 1	Model 2	Model 3	Model 1	Model 2	Model 3	Model 1	Model 2	Model 3
$\mu \cdot 10^3$	1.433 (0.435)	2.776 (0.413)	0.402 (0.291)	1.165 (0.683)	4.362 (0.520)	0.392 (0.312)	2.092 (0.435)	8.396 (1.290)	−0.008 (0.590)
$\omega_0 \cdot 10^6$	0.872 (0.630)	1.594 (0.678)	0.818 (0.616)	2.476 (1.155)	2.378 (0.885)	2.297 (1.080)	7.960 (3.932)	9.531 (2.758)	7.905 (1.266)
ω_1	0.108 (0.022)	0.270 (0.034)	0.111 (0.023)	0.151 (0.031)	0.309 (0.037)	0.144 (0.029)	0.065 (0.017)	0.133 (0.020)	0.064 (0.017)
ξ_1	0.898 (0.020)	0.759 (0.030)	0.898 (0.020)	0.843 (0.030)	0.703 (0.033)	0.849 (0.028)	0.911 (0.024)	0.827 (0.026)	0.912 (0.023)
$\overline{\alpha}$ or ν	1.075 (0.293)	1.570 (0.141)	4.889 (1.029)	2.818 (1.034)	2.220 (0.209)	9.870 (3.305)	4.736 (2.342)	2.672 (0.260)	14.011 (6.832)
$\overline{\beta}$ or ζ	−0.135 (0.067)	−0.481 (0.087)	0.904 (0.041)	−0.139 (0.130)	−0.795 (0.110)	0.896 (0.048)	−0.279 (0.320)	−1.008 (0.172)	0.941 (0.049)
$\ln L_1$	2382	2364	2381	2349	2335	2351	1991	1972	1991
RV	–	−1.602	−0.570	–	−1.328	0.848	–	−2.325*	0.116
$\ln L_2$	9724	9811	–	9511	9642	–	8355	8547	–
RV	–	3.467*	–	–	3.589*	–	–	5.366*	–
$LB(7)$	14.533*	17.297*	13.353	5.985	4.364	5.485	13.132	11.548	13.247
$LM(7)$	2.773	1.130	3.113	10.692	12.983	10.230	11.968	15.645*	12.941
AD	0.586	5.991*	1.439	0.796	4.505*	1.314	0.273	7.197*	0.434

Model 1 is the S&GARCH-NIG model for closing prices, Model 2 is the S&GARCH-NIG-HLC model, and Model 3 is the GARCH-skewed Student-t model. Standard errors are reported in parentheses. The parameters ν and ζ represent the degrees of freedom and the asymmetry of the skewed Student-t distribution, respectively. $\ln L_1$ and $\ln L_2$ are the logarithms of the likelihood function based solely on closing prices, and based on low, high and closing prices, respectively. RV is the Rivers-Vuong test for model selection, where comparisons are made using the S&GARCH-NIG model with parameters estimated based only on closing prices as the benchmark. LB, LM and AD are the Ljung–Box test for the presence of autocorrelation, the Engle test for the presence of the ARCH effect, and the Anderson–Darling goodness of fit test, respectively.

* Indicates that the null hypothesis is rejected at the 0.05 level.

Table A.3
Evaluation of the volatility forecasts for different volatility proxies, and the MSE and MAE criteria for the Athex Composite Share Price index at the beginning of the Greek debt crisis: year 2010.

Model	MSE	MAE
Realized variance		
Model 1	2.360	3.026
Model 2	1.599	2.090
Model 3	2.334	3.000
First-order autocorrelation-adjusted RV		
Model 1	2.692	3.247
Model 2	2.146	2.453
Model 3	2.665	3.228
Bi-power variation		
Model 1	1.723	2.945
Model 2	0.641	1.859
Model 3	1.706	2.924

Model 1 is the S&GARCH-NIG model for closing prices, Model 2 is the S&GARCH-NIG-HLC model, and Model 3 is the GARCH-skewed Student-t model. Volatility proxies for 15-min returns were employed. The values of the MSE and MAE were multiplied by 10^7 and 10^4, respectively.

References

Alizadeh, S., Brandt, M. W., & Diebold, F. X. (2002). Range-based estimation of stochastic volatility models. *Journal of Finance*, *57*(3), 1047–1091.

Andersen, T. G., Bollerslev, T., Diebold, F. X., & Labys, P. (2003). Modeling and forecasting realized volatility. *Econometrica*, *71*(2), 579–625.

Andersson, J. (2001). On the normal inverse Gaussian stochastic volatility model. *Journal of Business & Economic Statistics*, *19*(1), 44–54.

Arroyo, J., Espínola, R., & Maté, C. (2011). Different approaches to forecast interval time series: A comparison in finance. *Computational Economics*, *37*(2), 169–191.

Arroyo, J., González-Rivera, G., & Maté, C. (2010). Forecasting with interval and histogram data: Some financial applications. In A. Ullah, & D. E. A. Giles (Eds.), *Handbook of empirical economics and finance* (pp. 247–280). Chapman and Hall/CRC.

Barndorff-Nielsen, O. E. (1997). Normal inverse Gaussian distributions and stochastic volatility modelling. *Scandinavian Journal of Statistics*, *24*(1), 1–13.

Barndorff-Nielsen, O. E., & Shephard, N. (2001). Modelling by Lévy processes for financial econometrics. In O. E. Barndorff-Nielsen, T. Mikosch, & S. I. Resnick (Eds.), *Lévy processes—theory and applications* (pp. 283–318). Boston: Birkhäuser.

Barndorff-Nielsen, O. E., & Shephard, N. (2004). Power and bipower variation with stochastic volatility and jumps. *J. Financ. Econ.*, *2*(1), 1–37.

Brandt, M. W., & Jones, C. S. (2006). Volatility forecasting with range-based EGARCH models. *Journal of Business & Economic Statistics*, *24*, 470–486.

Chou, R. Y. (2005). Forecasting financial volatilities with extreme values: The conditional autoregressive range (CARR) model. *Journal of Money, Credit and Banking*, *37*(3), 561–582.

Chou, R. Y., Chou, H.-C., & Liu, N. (2010). Range volatility models and their applications in finance. In C.-F. Lee, A. C. Lee, & J. Lee (Eds.), *Handbook of quantitative finance and risk management* (pp. 1273–1281). US: Springer.

Cox, D. R., & Miller, M. D. (1965). *The theory of stochastic processes*. London: Methuen and Co.

Duan, J.C. (1999). Conditionally fat-tailed distributions and the volatility smile in options. Working Paper, Hong Kong University of Science and Technology, Hong Kong.

Engle, R. F., & Gallo, G. M. (2006). A multiple indicators model for volatility using intra-daily data. *Journal of Econometrics*, *131*(1–2), 3–27.

Fiszeder, P., & Perczak, G. (2013). A new look at variance estimation based on low, high and closing prices taking into account the drift. *Statistica Neerlandica*, *67*(4), 456–481.

Forsberg, L., & Bollerslev, T. (2002). Bridging the gap between the distribution of realized (ECU) volatility and ARCH modelling (of the euro): The GARCH-NIG Model. *Journal of Applied Econometrics*, *17*(5), 535–548.

Garman, M. B., & Klass, M. J. (1980). On the estimation of security price volatilities from historical data. *Journal of Business*, *53*(1), 67–78.

Hansen, P. R. (2005). A test for superior predictive ability. *Journal of Business & Economic Statistics*, *23*(4), 365–380.

Hansen, P. R., Huang, Z., & Shek, H. H. (2012). Realized GARCH: a joint model for returns and realized measures of volatility. *Journal of Applied Econometrics*, *27*(6), 877–906.
Hansen, P. R., & Lunde, A. (2006a). Consistent ranking of volatility models. *Journal of Econometrics*, *131*(1–2), 97–121.
Hansen, P. R., & Lunde, A. (2006b). Realized variance and market microstructure noise. *Journal of Business & Economic Statistics*, *24*(2), 127–161.
Hansen, P. R., Lunde, A., & Nason, J. M. (2011). The model confidence set. *Econometrica*, *79*(2), 453–497.
Jensen, M. B., & Lunde, A. (2001). The NIG-S&ARCH model: a fat tailed, stochastic, and autoregressive conditional heteroscedastic volatility model. *Econometrics Journal*, *4*, 319–342.
Li, H., & Hong, Y. (2011). Financial volatility forecasting with range-based autoregressive volatility model. *Finance Research Letters*, *8*(2), 69–76.
Lildholdt, P.M. (2002). Estimation of GARCH models based on open, close, high, and low prices. Working Paper Series 128. Centre for Analytical Finance, University of Aarhus, Aarhus School of Business.
Maia, A. L. S., & de Carvalho, F. A. T. (2011). Holt's exponential smoothing and neural network models for forecasting interval-valued time series. *International Journal of Forecasting*, *27*(3), 740–759.
Maia, A. L. S., de Carvalho, F. A. T., & Ludermir, T. B. (2008). Forecasting models for interval-valued time series. *Neurocomputing*, *71*(16–18), 3344–3352.
Osiewalski, J., & Pipień, M. (1999). Bayesian forecasting of exchange rates using GARCH models with skewed t conditional distributions. In W. Welfe (Ed.), *MACROMODELS'98—proceedings of the conference* (pp. 195–218). Łódź: Absolwent.
Parkinson, M. (1980). The extreme value method for estimating the variance of the rate of return. *Journal of Business*, *53*(1), 61–65.
Patton, A. J. (2011). Volatility forecast comparison using imperfect volatility proxies. *Journal of Econometrics*, *160*(1), 246–256.
Perczak, G., & Fiszeder, P. (2014). Model GARCH—wykorzystanie dodatkowych informacji o cenach minimalnych i maksymalnych. *Bank i Kredyt*, *45*(2), 105–132.
Pigorsch, C., Pigorsch, U., & Popov, I. (2012). Volatility estimation based on high-frequency data. In J.-C. Duan, W. K. Härdle, & J. E. Gentle (Eds.), *Springer handbooks of computational statistics*, *Handbook of computational finance* (pp. 335–369). Berlin, Heidelberg: Springer.
Rivers, D., & Vuong, Q. (2002). Model selection tests for nonlinear dynamic models. *The Econometrics Journal*, *5*(1), 1–39.
Rogers, L. C. G., & Satchell, S. E. (1991). Estimating variance from high, low and closing prices. *The Annals of Applied Probability*, *1*(4), 504–512.
Venter, J. H., De Jongh, P. J., & Griebenow, G. (2005). NIG-GARCH models based on open, close, high and low prices: Theory and methods. *South African Statistical Journal*, *39*(2), 79–101.
Venter, J. H., De Jongh, P. J., & Griebenow, G. (2006). GARCH-type volatility models based on Brownian inverse Gaussian intra-day return processes. *Journal of Risk*, *8*(4), 97–116.
Vuong, Q. H. (1989). Likelihood ratio tests for model selection and non-nested hypotheses. *Econometrica*, *57*(2), 307–333.
Yang, D., & Zhang, Q. (2000). Drift-independent volatility estimation based on high, low, open, and close prices. *Journal of Business*, *73*(3), 477–491.

Piotr Fiszeder is Professor at the Faculty of Economic Sciences and Management, Nicolaus Copernicus University in Toruń, Poland. He is a member of the Editorial Boards of several scientific journals, and author of over 60 academic publications in financial econometrics, including a book entitled ''The class of GARCH models in empirical finance research''. He has participated in many research projects, with support from the Ministry of Science and Higher Education in Poland and the European Community's Phare ACE Programme. At present, he is leading a research project entitled ''Modelling and Forecasting Volatility—Usage of Additional Information Contained in Low and High Prices'', financed by the National Science Centre.

Grzegorz Perczak has a Master of Science in Applied Mathematics and is a graduate of the Faculty of Fundamental Issues of Technique of Wroclaw University of Technology. He gained his early professional experience in several Polish banks while working as a member and head of the Market Analysis and Forecasts Department. Next, he worked as a fund manager of debt instruments in the Investment Fund Company, and later supervised the Option board at Brokerage House. Currently he is writing a doctoral thesis at the Faculty of Economics and Management of the Nicolaus Copernicus University in Toruń, taking part in a research project funded by the National Science Centre.