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Low and high prices can improve covariance forecasts: The evidence based on currency rates

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Abstract
In this paper we introduce a new specification of the BEKK model, where its parameters are estimated with the use of closing and additionally low and high prices. In an empirical application, we show that the use of additional information related to low and high prices in the formulation of the BEKK model improved the estimation of the covariance matrix of returns and increased the accuracy of covariance and variance forecasts based on this model, compared with using closing prices only. This analysis was performed for the following three most heavily traded currency pairs in the Forex market: EUR/USD, USD/JPY, and GBP/USD. The main result obtained in this study is robust to the applied forecast evaluation criterion. This issue is important from a practical viewpoint, because daily low and high prices are available with closing prices for most financial series.

KEYWORDS
covariance of returns, currencies, forecasting, low and high prices, volatility models

1 | INTRODUCTION

Volatility plays a key role in many financial and macroeconomic issues. Volatility models of financial instruments that are commonly used in practice are largely based solely on closing prices. However, the application of information about low and high prices may lead to much more accurate estimates of volatility. The outcomes of empirical and simulation studies show that variance estimators constructed based on low, high, and additionally open and closing prices are from more than five up to even more than seven times more efficient than estimators constructed exclusively on closing prices (see, e.g., Fiszeder & Perczak, 2013; Garman & Klass, 1980; Parkinson, 1980; Rogers & Satchell, 1991; Yang & Zhang, 2000). Despite good statistical properties, these estimators have not found widespread use in empirical studies, due to the fact of omission of the time dependence of variance. In recent years, however, numerous dynamic models have been constructed based on the price range, or its transformations, which is the difference between high and low prices (see, e.g., Alizadeh, Brandt, & Diebold, 2002; Brandt & Jones, 2006; Chou, 2005; Mapa, 2003; Molnar, 2011). Low and high prices were also applied to construct the likelihood function used for the estimation of parameters of generalized autoregressive conditional heteroskedasticity (GARCH) models (see Fiszeder & Perczak, 2016; Lildholdt, 2002; Venter, de Jongh, & Griebenow, 2005).

The models in all of the above-cited studies are for univariate processes. In financial applications, however, the use of univariate models rarely turns out to be sufficient. Investment portfolios consist of many assets whose returns are often related (in the mean or variance) and additionally have time-varying conditional variances. Analysis of the multivariate processes is therefore necessary for the construction and valuation of portfolios of financial instruments and the management of its risk. Analyses of multivariate models based on low and high prices are still at the initial stage of research. The idea of the construction of multivariate volatility models based on low and high prices consists in applying univariate model specifications based on the price range and incorporating them into multivariate models of the covariance matrix of returns (see Asai, 2013; Chou & Cai, 2009; Chou, Wu, & Liu, 2009; Su & Wu, 2014). In this paper, an alternative approach formulated on the basis of the estimator of the covariance of returns based on low and high prices (see Brandt & Diebold, 2006; Brunetti & Lildholdt, 2002; Fernandes, Mota, & Rocha, 2005) and the BEKK model (Baba, Engle, Kraft, & Kroner, 1990; Engle & Kroner, 1995) is applied to describe currency returns. The BEKK model was chosen because it is a very popular multivariate GARCH model, often used in empirical finance, and the application of the covariance estimator based on low and high prices is the most intuitive for this model and it does not require additional transformations.

This study offers two main contributions. The first one is a proposition of a new specification of the BEKK model, where its parameters are estimated with the use of closing and additionally low and high prices. The statistical properties of the new model are the same as the traditional BEKK model, where parameters are estimated based only on closing prices. The second contribution is to show that the use of additional information related to low and high prices in the formulation of the BEKK model can improve the estimation of the covariance matrix of returns and increase the accuracy of covariance and variance forecasts based on this model, compared with using closing prices only. To the best of our knowledge, this is the first attempt in the literature to demonstrate the superiority of this approach for forecasting. This issue is important from a practical viewpoint, because daily low and high prices are almost always commonly available with closing prices for financial series.

The remainder of this paper is organized as follows. Section 2 provides a description of applied models and methods. In Section 3 relations between the three most heavily traded currency pairs in the Forex market, namely EUR/USD, USD/JPY, and GBP/USD, are analyzed. Section 4 evaluates the forecasts of the covariance matrix of returns for the proposed and competing models. Section 5 provides conclusions.

2 | THE BEKK MODEL WITH LOW, HIGH, AND CLOSING PRICES

The multivariate GARCH models are one of the most popular classes of models to describe financial time series (see, e.g., Bauwens, Hafner, & Laurent, 2012). The general form of a multivariate GARCH model is the VECH model[1] (Kraft & Engle, 1983). The estimation of its parameters is, however, very difficult even for a small number of assets. For this reason, simpler parametrizations of multivariate GARCH models were introduced in the literature; among them, the BEKK[2] (Baba et al., 1990; Engle & Kroner, 1995) and DCC (dynamic conditional correlations, Engle, 2002) models are very popular.

Let us assume that the $\boldsymbol{\varepsilon}_t$ ($N\times 1$ vector) is the multivariate innovation process for the conditional mean (or in a particular case the multivariate return process) and can be written as

$$\boldsymbol{\varepsilon}_t \mid \psi_{t-1} \sim \mathrm{D}(0, \mathbf{cov}_t), \tag{1}$$

where ψ_{t-1} is the set of all information available at time $t-1$, D is the conditional multivariate density function, and $\mathbf{cov}_t$ is an $N\times N$ symmetric conditional covariance matrix.

A popular specification of the multivariate GARCH model is the BEKK(p, q) model. It can be written as

$$\mathbf{cov}_t = \mathbf{CC} + \sum_{i=1}^{q} \mathbf{D}_i \boldsymbol{\varepsilon}_{t-i} \boldsymbol{\varepsilon}_{t-i}' \mathbf{D}_i' + \sum_{j=1}^{p} \mathbf{E}_j \, \mathbf{cov}_{t-j} \mathbf{E}_j', \tag{2}$$

where $\mathbf{C}$, $\mathbf{D}_i$ and $\mathbf{E}_j$ are $N\times N$ parameter matrices and $\mathbf{C}$ is an upper triangular matrix.

The advantages of this formulation are a positive definiteness of $\mathbf{cov}_t$ and the ability to describe time-varying conditional correlations between the returns and the cross dynamics of conditional covariances. It is a more complex specification than the DCC model, that is, another very popular multivariate GARCH model. The correlations based on the DCC model are restricted to have a very similar dynamic structure for all assets. The parameters of the DCC model can be estimated consistently in two steps, which makes this approach relatively simple and possible to apply even when N is high. In contrast, the fully parametrized BEKK model is feasible only for small values of N: typically less than 10.

We apply the estimator of the covariance of returns based on low and high prices. The intuition is that such an estimator is better than the one based on only closing prices (see Brunetti & Lildholdt, 2002). Let us assume that the two assets have a very similar daily pattern (i.e., they move exactly in the same direction throughout the day). This implies a high daily covariance between the assets, which can be captured by the estimator based

[1]The name of the model comes from the application of the vech operator for the conditional covariance matrix.

[2]The name of the model is formed by the first letters of the authors' surnames.

on low and high prices. If the daily return calculated on the basis of closing prices is close to zero for both assets, than the estimate of covariance based on daily returns will unfortunately fail to capture the daily comovements.

The idea of the applied approach considered by Brunetti and Lildholdt (2002), Fernandes et al. (2005), and Brandt and Diebold (2006) is based on the transformed formula for the variance of the sum of two random variables. The covariance between them can be expressed as

$$\operatorname{cov}(X, Y) = [\operatorname{var}(X+Y) - \operatorname{var}(X) - \operatorname{var}(Y)]/2, \quad (3)$$

where all variances are estimated with the use of low and high prices. Different kinds of estimators based on daily low, high, or additionally open and closing prices, like Garman and Klass (1980), Parkinson (1980), or Rogers and Satchell (1991), can be applied in the above formula. Brunetti and Lildholdt (2002) used the Parkinson estimator and termed the estimator of covariance as the co-range.

The considered method of covariance estimation is possible when the range of a portfolio of assets is given (the variance for the sum of variables X and Y is needed). Such data are unfortunately rarely available (see, for some examples, Brandt & Diebold, 2006). However, the range of a portfolio of assets can be calculated based on intraday data (tick-by-tick data). The realized covariance, that is, the estimator of covariance constructed based on intraday prices, is more efficient than the estimator based on high and low prices, but the latter can be less sensitive to some sources of the microstructure noise arising from the bid–ask spread and nonsynchronous trading (see Monte Carlo simulations by Brandt & Diebold, 2006).

The range of a portfolio return can be calculated without the use of intraday data in the case of foreign exchange rates. Consider two exchange rates of currencies A and B in terms of currency C, denoted by A/C and B/C. In the absence of triangular arbitrage, the cross-rate can be given as

$$\Delta\ln \mathrm{A/B} = \Delta\ln \mathrm{A/C} - \Delta\ln \mathrm{B/C}. \quad (4)$$

The estimator of the covariance of returns is then expressed as:

$$\operatorname{cov}(\Delta\ln \mathrm{A/C}, \Delta\ln \mathrm{B/C}) = [\operatorname{var}(\Delta\ln \mathrm{A/C}) + \operatorname{var}(\Delta\ln \mathrm{B/C}) - \operatorname{var}(\Delta\ln \mathrm{A/B})]/2. \quad (5)$$

The idea of using triangular arbitrage in order to calculate the covariance of returns has been employed, among others, by Lopez and Walter (2001) for the implied covariance, in Brunetti and Lildholdt (2002) for the co-range, and in Andersen, Bollerslev, Diebold, and Labys (2003) for the realized covariance.

Monte Carlo simulations performed by Brunetti and Lildholdt (2002) indicate that the estimator of covariance based on low and high prices is biased (downward for positive correlation and upward for negative correlation) but highly efficient (approximately five times more efficient than the estimator based on closing prices when the Parkinson estimator is applied).

We propose the new formulation of the BEKK model with the usage of low and high prices (denoted by BEKK-HL(p, q)), which can be expressed as

$$\mathbf{cov}_t = \mathbf{KK} + \sum_{i=1}^{q} \mathbf{L}_i \mathbf{G}_{t-i} \mathbf{L}_i' + \sum_{j=1}^{p} \mathbf{M}_j \, \mathbf{cov}_{t-j} \mathbf{M}_j', \quad (6)$$

where $\mathbf{K}$, $\mathbf{L}_i$ and $\mathbf{M}_j$ are $N \times N$ parameter matrices and $\mathbf{K}$ is an upper triangular matrix, and $\mathbf{G}_{t-i}$ are covariance matrices of returns at time $t-i$ calculated with the use of low and high prices. Variances of returns formulated with low and high prices are on diagonals of matrices $\mathbf{G}_{t-i}$, while covariances of returns based on Equation 3 are outside the diagonal. In comparison to the traditional BEKK model, instead of variances and covariances of returns calculated on the basis of closing prices, we apply more efficient estimators of variances and covariances formulated with the use of low and high prices. The statistical properties of the new model are the same as the BEKK model, where parameters are estimated based only on closing prices.

Different kinds of estimators based on daily low, high, or additionally open and closing prices can be applied in matrices $\mathbf{G}_{t-i}$. In the empirical application in the subsequent sections the Parkinson estimator (Parkinson, 1980) was used. It can be expressed as

$$\sigma_{tP}^2 = \frac{[\ln(H_t/L_t)]^2}{4\ln 2}, \quad (7)$$

where H_t and L_t are daily high and low prices.

It should be noted that the proposed model is parsimonious and there are no additional parameters relative to the model based only on returns of closing prices. Of course, the parsimony refers only to the traditional BEKK model, because both models suffer from the so-called curse of dimensionality. Parameters of both the BEKK and BEKK-HL models can be estimated by maximum likelihood or quasi-maximum likelihood methods (see Comte & Lieberman, 2003, for properties of these estimators).

3 | MODELING CURRENCY RATES: EUR/USD, USD/JPY AND GBP/USD

The usefulness of the proposed model is illustrated by the study of the three most heavily traded currency pairs in the Forex market, namely EUR/USD, USD/JPY and GBP/USD. First, a valuation of the models considered was performed for daily data for the 11-year period from January 2, 2006 to December 30, 2016 (2,853 returns). The descriptive statistics for the percentage returns calculated as $r_t = 100 \ln (p_t/p_{t-1})$, where p_t is the closing price at time t, are presented in Table 1.

The variability of returns, measured by the standard deviation, was quite similar for all currency pairs, but significant differences were found in the skewness and kurtosis of the distributions. Due to Brexit, the distribution of returns was more leptokurtic and the minimum return was significantly lower for GBP/USD than for the remaining pairs.

The reference model is the traditional BEKK model (Equation 2), where parameters are estimated based only on closing prices. The second model is the proposed BEKK-HL model (Equation 6), where parameters are estimated based on low, high, and closing prices. A natural competitor for both BEKK models is the DCC model, where parameters are estimated based on closing prices. The DCC model is used because it is a less complex parametrization and it is much easier to estimate its parameters. The multivariate Student t-distribution is employed as a conditional multivariate density function in Equation 1 for the three models as a way of providing a better description of the fat tails of the distribution of considered exchange rates. The Student t-distribution is used most frequently as the conditional distribution in empirical applications for financial series.

In the BEKK-HL model the Parkinson estimator (Equation 7) was applied to calculate variances in Equation 3. It was also used and advocated by Brunetti and Lildholdt (2002) and Brandt and Diebold (2006). The main conclusions of our study do not change, however, for other estimators, such as, Garman and Klass (1980) or Rogers and Satchell (1991).

The considered exchange rates were not cointegrated (according to the Johansen test) and there were no constant relations in the conditional means of returns (according to the VAR model), which is why there are only constants in the conditional mean equations of returns. The parameters of these models are estimated using the maximum likelihood method. The results of the estimations are presented in Table 2.

The logarithms of the likelihood function for the three considered models are based solely on closing prices (indicated by ln L). The Rivers and Vuong test (RV; Rivers & Vuong, 2002) was also performed, which allowed verification of the hypothesis that the likelihood functions of two nonnested competing models are asymptotically equivalent. The RV test is a generalization of the Vuong tests (Vuong, 1989), which can be applied to nonlinear models of time series. According to the test, the BEKK-HL model performed significantly better than the competing models based solely on closing prices. Moreover, there were no significant differences between the BEKK and DCC models estimated with the use of closing prices. The Bayesian information criterion also pointed at the BEKK-HL model as superior.

It is interesting to compare the estimates of parameters between the BEKK and BEKK-HL models. The application of low and high prices to estimation changed the estimates of the BEKK model parameters significantly. Specifically, the estimates of the parameters l_{ii} were higher than the estimates of the parameters d_{ii}, while the estimates of the parameters m_{ii} were lower than the estimates of the parameters e_{ii}. For different scenarios of returns one can easily show that shocks in the previous period have a stronger impact on current covariances and variances, and thus the model with parameters estimated based on low, high, and closing prices has a faster response to changing market conditions. This is important in terms of both the modeling and forecasting returns, because a slow response to abrupt changes in the market is widely cited as one of the greatest weaknesses of GARCH-type models formulated based on closing prices (e.g., Andersen et al., 2003; Hansen, Huang, & Shek, 2012).

4 | FORECASTING COVARIANCES AND VARIANCES OF RETURNS

The main purpose of this empirical study was to compare the forecasting performance of the BEKK model estimated based on low, high, and closing prices with that

TABLE 1 Summary statistics of daily returns for currency pairs

Exchange rates	Mean $\times 10^5$	Minimum	Maximum	Standard deviation	Skewness	Excess kurtosis
EUR/USD	−4.1738	−0.0255	0.0350	0.0062	0.0876	2.0347*
JPY/USD	0.2636	−0.0545	0.0378	0.0067	0.0511	4.5491*
GBP/USD	−11.7254	−0.0832	0.0287	0.0061	−1.1976*	14.041*

Note. Asterisk indicates that the null hypothesis (the skewness or excess kurtosis is equal to zero) was rejected at the 0.05 level.

TABLE 2 Results estimated for the three multivariate GARCH models

	BEKK			BEKK-HL			DCC	
Parameter	Estimate	SE	Parameter	Estimate	SE	Parameter	Estimate	SE
γ_{01}	0.0047	0.0086	γ_{01}	0.0005	0.0117	γ_{01}	0.0022	0.0092
γ_{02}	−0.0152	0.0090	γ_{02}	−0.0091	0.0125	α_{01}	0.0005	0.0005
γ_{03}	0.0066	0.0083	γ_{03}	−0.0008	0.0106	α_{11}	0.0401	0.0057
c_{11}	−0.0286	0.0089	k_{11}	−0.0101	0.0180	β_{11}	0.9588	0.0056
c_{21}	−0.0104	0.0213	k_{21}	0.0314	0.0732	ν	9.6468	1.5830
c_{22}	0.0674	0.0092	k_{22}	0.0585	0.1433	γ_{02}	−0.0117	0.0094
c_{31}	−0.0085	0.0160	k_{31}	0.0179	0.0259	α_{02}	0.0034	0.0015
c_{32}	−0.0183	0.0089	k_{32}	−0.0284	0.0365	α_{12}	0.0596	0.0104
c_{33}	0.0332	0.0077	k_{33}	0.0021	0.1009	β_{12}	0.9362	0.0109
d_{11}	0.1874	0.0180	l_{11}	0.2678	0.0370	ν	5.0514	0.4775
d_{12}	−0.0314	0.0132	l_{12}	−0.0636	0.0220	γ_{03}	0.0034	0.0088
d_{13}	0.0083	0.0194	l_{13}	0.0086	0.0376	α_{03}	0.0016	0.0009
d_{21}	0.0189	0.0209	l_{21}	0.0259	0.0439	α_{13}	0.0458	0.0080
d_{22}	0.2350	0.0177	l_{22}	0.3074	0.0322	β_{13}	0.9503	0.0084
d_{23}	−0.0549	0.0242	l_{23}	−0.0741	0.0472	ν	9.3613	2.0626
d_{31}	−0.0205	0.0253	l_{31}	−0.0101	0.0352	α	0.0278	0.0042
d_{32}	−0.0506	0.0150	l_{32}	−0.0670	0.0222	β	0.9351	0.0122
d_{33}	0.2090	0.0205	l_{33}	0.2658	0.0322	ν	4.9512	0.0840
e_{11}	0.9825	0.0037	m_{11}	0.9616	0.0110			
e_{12}	0.0087	0.0037	m_{12}	0.0240	0.0094			
e_{13}	−0.0042	0.0044	m_{13}	−0.0056	0.0130			
e_{21}	−0.0034	0.0048	m_{21}	−0.0003	0.0144			
e_{22}	0.9650	0.0051	m_{22}	0.9403	0.0137			
e_{23}	0.0139	0.0063	m_{23}	0.0161	0.0155			
e_{31}	0.0054	0.0058	m_{31}	0.0022	0.0118			
e_{32}	0.0149	0.0047	m_{32}	0.0287	0.0100			
e_{33}	0.9728	0.0053	m_{33}	0.9569	0.0105			
ν	6.7657	0.4632	ν	7.2524	0.8290			
ln L	−6,162.20		ln L	−6,087.19		ln L	−6,179.16	
RV	–		RV	3.0246*		RV	1.5784	
BIC	12,547		BIC	12,397		BIC	12,494	

Note. In all models the currency pairs are applied in the following order: EUR/USD, JPY/USD, and GBP/USD. γ_{01}, γ_{02}, γ_{03} are constants; c_{ij}, d_{ij}, e_{ij}, and k_{ij}, l_{ij}, m_{ij} are parameters in the matrices $\mathbf{C}$, $\mathbf{D}_1$, $\mathbf{E}_1$ (Equation 2), and $\mathbf{K}$, $\mathbf{L}_1$, $\mathbf{M}_1$ (Equation 6), respectively; α_{0i}, α_{1i}, β_{1i} are parameters of the univariate GARCH models; α, β are parameters of the DCC model; the parameter ν represents the degrees of freedom of the multivariate Student t-distribution. ln L is the logarithm of the likelihood function. RV is the Rivers–Vuong test statistic for model selection, where comparisons were made with the BEKK model, for which the parameters were estimated based only on closing prices as the benchmark. BIC is the Bayesian information criterion. Asterisk indicates that the null hypothesis was rejected at the 0.05 level.

of the model based only on closing prices. Out-of-sample 1-day-ahead forecasts of the covariance and variance were formulated based on the models, where parameters were estimated separately each day based on a rolling sample with a fixed size of 500 (approximately a 2-year period; the first in-sample period was from January 3, 2006 to December 31, 2007). The evaluation of forecasts was performed for the 9-year period from January 2, 2008 to December 30, 2016. It is a relatively long period which covers both turbulent periods, such as the global financial

TABLE 3 The evaluation of the covariance forecasts: the MSE and MAE criteria

	Forecast evaluation criteria					
Model	**MSE**	**SPA *p*-value**	**MCS *p*-value**	**MAE**	**SPA *p*-value**	**MCS *p*-value**
EUR/USD, JPY/USD						
BEKK closing prices	0.089	—	0.004	0.138	—	0.000
BEKK-HL	0.085	0.002	1.000*	0.130	0.000	1.000*
DCC closing prices	0.101	0.536	0.001	0.139	0.654	0.000
EUR/USD, GBP/USD						
BEKK closing prices	0.120	—	0.245*	0.135	—	0.000
BEKK-HL	0.116	0.049	1.000*	0.126	0.000	1.000*
DCC closing prices	0.118	0.022	0.336*	0.129	0.000	0.035
JPY/USD, GBP/USD						
BEKK closing prices	0.206	—	0.095	0.122	—	0.000
BEKK-HL	0.201	0.029	1.000*	0.112	0.000	1.000*
DCC closing prices	0.209	0.686	0.094	0.121	0.253	0.001

Note. Realized covariance is used as a proxy of covariance and estimated as the sum of products of 15-min returns. The SPA test is performed for pairs of models with the BEKK for closing prices used as a benchmark. The MCS test is performed for the three models jointly. Asterisk indicates that models belong to the MCS with a confidence level of 0.90.

crisis of 2008, the European sovereign debt crisis, or Brexit, but also tranquil periods, therefore, the results should be robust to the state of the global economy.

As a proxy of the daily covariance for the evaluation of forecasts, the sum of products of intraday returns (the realized covariance) was employed, while as a proxy of the daily variance the sum of squared intraday returns (the realized variance) was used. One significant problem when using such data is the choice of appropriate frequency of observations (see, e.g., Pigorsch, Pigorsch, & Popov, 2012). In this study 15-min returns are applied but the main results do not change for 5- or 30-min returns. The forecasts of the models were evaluated based on two primary measures, namely the mean squared error (MSE) and the mean absolute error (MAE). The MSE is the criterion that is used most frequently in empirical studies, and is also robust to the use of a noisy volatility proxy (it yields the same ranking of competing forecasts using an unbiased volatility proxy; see Hansen & Lunde, 2006; Patton, 2011). Whereas the MAE is less sensitive to outliers. In order to evaluate the statistical significance of the results two different tests were applied: the test of superior predictive ability (SPA) of Hansen (2005) and the model confidence set (MCS) of Hansen, Lunde, and Nason (2011). In the first approach alternative forecasts are compared with a benchmark forecast. In this study, a pairwise comparison was performed and results are presented with respect to the BEKK model based on closing prices used as a benchmark. On the other hand, however, the MCS procedure does not require a benchmark model to be specified. The MCS contains the best forecasting models with a certain probability. The results of the study for covariance and variance forecasts are presented in Tables 3 and 4, respectively.

According to the results of the SPA test for the MSE and MAE criteria (the outcomes in Table 3, but also the results not presented to save space, when the DCC model is used as a benchmark), the covariance forecasts from the models where parameters were estimated based only on closing prices were inferior to the forecasts from the model where parameters where estimated on the basis of low, high, and closing prices (at the 5% significance level). The same conclusion results from the MCS test. The only exception is the relation between the EUR/USD and GBP/USD pairs under the MSE measure for which all three models belonged to the MCS, and there was no evidence to reject the null hypothesis of equal predictive ability.

The values of MSE were the lowest for the covariance between the EUR/USD and JPY/USD rates, and the highest for the relation between the JPY/USD and GBP/USD pairs. The opposite direction of errors was under the MAE measure. This means that forecasting the covariance between the JPY/USD and GBP/USD rates was the most difficult task. However, it was mainly connected to outliers, and when less weight was assigned to outliers then the forecasting errors were the lowest for those pairs.

The results of the SPA test for the MSE and MAE criteria (the outcomes in Table 4, but also the results not presented to save space, when the DCC model was used as a benchmark) indicate that the variance forecasts from the model where parameters were estimated based only on closing prices were inferior to the forecasts from the

TABLE 4 Evaluation of variance forecasts: MSE and MAE criteria

	Forecast evaluation criteria					
Model	**MSE**	**SPA *p*-value**	**MCS *p*-value**	**MAE**	**SPA *p*-value**	**MCS *p*-value**
EUR/USD						
BEKK closing prices	0.171	—	0.003	0.217	—	0.000
BEKK-HL	0.157	0.001	1.000*	0.200	0.000	1.000*
DCC closing prices	0.167	0.000	0.014	0.205	0.000	0.002
JPY/USD						
BEKK closing prices	0.618	—	0.161*	0.296	—	0.000
BEKK-HL	0.596	0.049	1.000*	0.277	0.000	1.000*
DCC closing prices	0.631	0.949	0.079	0.300	0.973	0.000
GBP/USD						
BEKK closing prices	1.032	—	1.000*	0.213	—	0.006
BEKK-HL	1.043	0.709	0.632*	0.203	0.005	1.000 *
DCC closing prices	1.085	0.924	0.322*	0.214	0.589	0.001

Note. Realized variance is used as a proxy of volatility and estimated as the sum of squared 15-min returns. The SPA test is performed for pairs of models with the BEKK for closing prices used as a benchmark. The MCS test is performed for the three models jointly. Asterisk indicates that models belong to the MCS with a confidence level of 0.90.

TABLE 5 Evaluation of covariance and variance forecasts: coefficient of determination

	Covariances			Variances		
Model	**EUR/USD, JPY/USD**	**EUR/USD, GBP/USD**	**JPY/USD, GBP/USD**	**EUR/USD**	**JPY/USD**	**GBP/USD**
BEKK closing prices	0.232	0.358	0.104	0.359	0.204	0.145
BEKK-HL	0.266	0.381	0.126	0.410	0.234	0.152
DCC closing prices	0.127	0.368	0.071	0.374	0.198	0.121

Note. Realized covariance and realized variance are based on 15-min returns used as a proxy of covariance and variance, respectively.

models where parameters where estimated on the basis of low, high, and closing prices. The only exception was the GBP/USD exchange rate under the MSE measure, for which there were no significant differences between the forecasts of considered models. According to the results of the MCS test for the MSE criterion, only the BEKK-HL model belonged to the MCS for the EUR/USD pair, both BEKK models belonged to the MCS for the JPY/USD exchange rate, and all three models belonged to the MCS for the GBP/USD pair, while the results of the MCS test for MAE measure indicate BEKK-HL as the best forecasting model for all three exchange rates.

It is worth emphasizing that exceptions from the forecasting superiority of the model based on low and high prices, both for covariance and variance, took place under the MSE measure and did not occur under the MAE, which is a less sensitive measure to outliers. It should also be noted that the forecasting errors were significantly lower for the evaluation of covariance than variance.

Under the MSE measure, the lowest errors of the volatility forecasts were for the EUR/USD rate. Considerably higher errors were for the JPY/USD pair and the highest for the GBP/USD rate. Under the MAE criterion, the errors were significantly lower for the GBP/USD pair, which indicates that the difficulty in volatility forecasting for this rate was mainly caused by outliers which took place, for example, after Brexit.

It is also interesting to compare the forecasting errors of volatility for exchange rates with the errors for stock indices like S&P 500 or FTSE 100 (see Fiszeder & Perczak, 2016). The forecasting errors for equities were significantly higher than for currencies.

Other loss functions were also considered, but yielded similar results. Thus, to save space, Table 5 presents only the R^2 values from the Mincer–Zarnowitz regression.

For all covariances and variances the highest R^2 values were obtained for the BEKK-HL model and pointed again at this model as superior.

5 | CONCLUSIONS

Low and high prices deliver important information about volatility of financial assets. It is now commonly known that the usage of such prices in volatility models improves the volatility estimation and increases the accuracy of the volatility forecasts compared with models based only on closing prices. In recent years, in some papers it has been shown that the application of low and high prices can be equally beneficial also in the estimation and forecasting of covariance of returns.

In this study we have proposed a new specification of the BEKK model, where its parameters are estimated with the use of closing and additionally daily low and high prices. We have also presented an empirical application to the three most heavily traded currency pairs in the Forex market, namely EUR/USD, USD/JPY, and GBP/USD. We used additional information on low and high prices in the formulation of the BEKK model, which improved the estimation of the covariance matrix of returns and increased the accuracy of covariance and variance forecasts based on this model, compared with using closing prices only. This result was robust to the forecast evaluation criterion employed. In future, this method could be extended to other multivariate GARCH models, as well as to other volatility models such as the multivariate stochastic volatility models.

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