Bayesian analysis of long memory and persistence using ARFIMA models with an application to Polish stock market.

1. Introduction

This article describes Bayesian analysis of autoregressive fractionally integrated moving average (ARFIMA) models and provides an application to the weekly returns of the trust fund Pioneer 1 on Polish stock market. This paper also discusses basic concepts of long memory and persistence, associated with ARFIMA models.

The origin of interest in long memory processes has come from hydrology and climatology. The most known example is article by Hurst (1951). In econometrics, long memory models have really been used since around 1980 (Granger (1980), Granger, Joyeux (1980) and Hosking (1981)).

ARFIMA models have been used, for instance, in studying real output (Diebold and Rudebush (1989); Sowell (1992b)), foreign-exchange rates (Cheung (1993) and Peters (1991)), inflation (Delgado and Robinson (1994)). Bayesian analysis of ARFIMA models has been discussed by Koop, Ley, Osiewalski, Steel (1997) and Pai and Ravinshanker (1996).

The paper is organised as follows: The section 2 describes long memory processes and ARFIMA models. Section 2 also describes impulse response functions. Section 2 introduces Bayesian analysis of ARFIMA models. Section 4 presents application to the weekly returns of the trust fund Pioneer 1 on Polish stock market. Section 5 concludes.

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2. Long memory processes and ARFIMA models.

Most of the work in time series analysis has been concerned with series having the property that autocorrelation function decays exponentially to zero (e.g., ARMA processes), and then

$$\sum_{k=\infty}^{\infty} |\rho(k)| < \infty, \quad (2.1)$$

where \( \rho(k) \) is autocorrelation function.

Consider, for example, first-order autoregressive process

$$y_t = ay_{t-1} + \varepsilon_t, \quad a \in (-1,1), \quad (2.2)$$

where \( \varepsilon_t \) is a sequence of independent normal variables with mean zero and variance \( \sigma^2 \), \( \sum_{k=-\infty}^{\infty} \rho(k) \) is finite and equal to

$$\sum_{k=-\infty}^{\infty} a^{|k|} = 1 + \frac{2a}{1-a} < \infty. \quad (2.3)$$

It turns out however that for some data sets, condition 2.1 does not hold. These processes imply autocovariances that decay hyperbolically as the lag increases, and a spectral density \( f(\omega) \) has a pole at zero. Therefore autocorrelation function takes the form

$$\rho(k) = c_\rho |k|^{-\alpha}, \quad (2.4)$$

where \( k \) tends to infinity and \( c_\rho \) is a finite positive constant. As \( \alpha \in (0,1) \), this implies that

$$\sum_{k=-\infty}^{\infty} \rho(k) = \infty \quad (2.5)$$
Processes satisfied condition 2.5 are called *long memory processes*.

An example of a long memory process is so-called „fractional noise” (ARFIMA \((0, d, 0)\)) with autocovariances

\[
\gamma_k = 0.5\gamma_0 \left\{ k+1 \right\}^{2d+1} - 2\left\{ k \right\}^{2d+1} + \left\{ k-1 \right\}^{2d-1}, \quad \text{for } k = 1, 2, \ldots \tag{2.6}
\]

For \(d\) from interval \((0, 0.5)\) autocorrelation function is given by

\[
\gamma_k = ck^{2d-1}, \quad \text{as } k \to \infty \tag{2.7}
\]

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There are also processes with intermediate memory for \(\alpha > 1\) and hence

\[
\sum_{k=-\infty}^{\infty} |\rho(k)| < \infty.
\]
where \( c \) satisfying \( 0 < c < \infty \). Therefore expression 2.7 satisfies conditions of long memory (equations 2.4 and 2.5).

When \( 0 < d < 0.5 \), the ARFIMA \((0, d, 0)\) process is a process with long memory. The correlations and partial correlations decay monotonically and hyperbolically to zero as the lag increases. The spectral density is concentrated at low frequencies.

When \(-0.5 < d < 0\), the ARFIMA \((0, d, 0)\) process has an intermediate memory.

When \( d = 0 \), ARFIMA \((0, d, 0)\) is white noise.

A wider class of parametric models are the autoregressive fractionally integrated moving averages (ARFIMA \((p, d, q)\)), given by

\[
\phi(B)(1 - B)^d (x_t - \mu) = \theta(B) \epsilon_t ,
\]

where \( d > -1 \), \( \epsilon_t \) is white noise, \( B \) is the backward shift operator, \( \phi(\cdot) \) and \( \theta(\cdot) \) are polynomials of degrees \( p \) and \( q \) respectively. Therefore ARFIMA \((p, d, q)\) models are a generalisation of Box and Jenkins (1976) ARIMA models.

The reason for choosing this class of processes is that the effect of the autoregressive and moving average components decay exponentially, while the effect of the parameter \( d \) decays hyperbolically. Thus ARFIMA \((p, d, q)\) models describe the long-, medium-, and short-term behaviour.

Based on Koop, Ley, Osiewalski, Steel (1997) we can consider ARFIMA models for deviations from a deterministic trend, mainly

\[
\phi(B)(1 - B)^\delta z_t = \theta(B) \epsilon_t ,
\]

where

\[
z_t = (1 - B)(y_t - \mu t - \alpha) = \Delta y_t - \mu ,
\]

\[
u_t = y_t - \mu t - \alpha .
\]

Using 2.10 to substitute for \( z_t \) in 2.9 we obtain

\[
\phi(B)(1 - B)^d (y_t - \mu t - \alpha) = \theta(B) \epsilon_t ,
\]

where \( d = \delta + 1 \).

When
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\( \delta \in (-1, -0.5) \) the process \( y_t \) is a trend stationary process with long memory.

\( \delta \in (-0.5, 0.5) \) the process \( \Delta y_t \) is a stationary process with long memory for \( \delta > 0 \) and with intermediate memory for \( \delta < 0 \),

\( \delta = 0 \) the process \( y_t \) corresponds to the standard ARIMA \((p, 1, q)\) model.

The impulse response function (Campbell and Mankiw (1987); Diebold and Rudebusch (1989); Hauser, Pötcher, Reschnhofer (1992)) of the estimated ARFIMA models is one measure of the persistence in various macroeconomic time series.

Let assume that \( y_t \) is a process with stationary first differences \( \Delta y_t \)

\[ \Delta y_t = \mu + A(B) \epsilon_t = \mu + (1 + \alpha_1 B + \alpha_2 B^2 + \ldots) \epsilon_t \quad (2.13) \]

Impulse responses for a stationary process are the coefficients of its moving average representation (Koop et al. (1997)). An impulse response function, \( I(n) \) can be interpreted as the effect of a shock of size one at time \( t \) on \( y_{t+n} \),

\[ I(n) = 1 + \alpha_1 + \ldots + \alpha_n . \]

When the process \( y_t \) is stationary or stationary around a deterministic linear trend the effect of any shock is transitory,

in case of the unit root, the effect of any shock is permanent and equal to

\[ I_{t+\infty} = \theta(1)/\phi(1) \quad (2.14) \]

For instance, for a random walk \( y_t = y_{t-1} + \epsilon_t \) the effect of any shock is permanent and equal to one.

For ARFIMA models \( \phi(B)(1-B)^\delta (\Delta y_t - \mu) = \theta(B) \epsilon_t \) the impulse response function takes the form

\[ \Delta y_t = (1-B)^{-\delta} \theta(B)/\phi(B) \epsilon_t \quad (2.15) \]

Depending on parameter \( \delta \) the impulse response function satisfies

\[ I_{t+\infty} = \begin{cases} 
\infty & \text{for } \delta > 0 \\
\theta(1)/\phi(1) & \text{for } \delta = 0 \\
0 & \text{for } \delta < 0 
\end{cases} \quad (2.16) \]
Note that $\delta = 0$ corresponds to the standard ARIMA ($p$, $1$, $q$) model for $y_t$ and to the ARMA ($p$, $q$) model for $\Delta y_t$. Additionally, instead of examining the impulse response function at infinity $I(\infty)$, a ‘short-run’, ‘medium-run’ and ‘long-run’ effects of a shock can be considered, (e.g., $n = 4, 12, 40$)

3. Bayesian analysis of ARFIMA models.

Following Koop, Ley, Osiewalski, Steel (1997) prior density is given by

$$p(\omega, \mu, \sigma^{-2}) = p(\omega)p(\mu)p(\sigma^{-2}) \propto \sigma^{-2} p(\omega)$$  \hspace{1cm} (3.1)

where $p(\omega)$ is a uniform density truncated to the stationarity and invertibility region $\Omega$. The likelihood function based on $t$ observations is

$$p(w/\omega, \mu, \sigma^{-2}) = f_N^t (w/\mu_l, \sigma^2V)$$  \hspace{1cm} (3.2)

where $f_N^t$ is the $t$-variate Normal density function $w' = (\Delta y_1, ..., \Delta y_T)$, $\omega' = (\delta, \Theta, \Phi) \in \Omega$, $\Phi = (\phi_1, ..., \phi_p) \in C_p$, $\Theta = (\theta_1, ..., \theta_q) \in C_q$, $\Omega = (-1, 0, 5) \times C_q \times C_p$, $\mu \in R$, $\sigma^{-2} \in R_+$, $l_t$ - $t \times 1$ vector of ones, $V$ - matrix with elements $v_{i,j} = \sigma^{-2} \gamma(i-j)$ for $i,j = 1, ..., T$; $\gamma(s)$ is the autocovariance function given in Sowell (1992a). After combining 3.1 with 3.2 and integrating out $\mu, \sigma^{-2}$, the marginal posterior pdf is

$$p(\omega / \text{dane}) = K^{-1} V^{-1/2} [V^{-1} l_t]^{-1/2} SSE^{-l/2} p(\omega),$$  \hspace{1cm} (3.3)

where

$$K = \int_{\Omega} V^{-1/2} [V^{-1} l_t]^{-1/2} SSE^{-l/2} p(\omega) d\omega,$$

$$SSE = (w - \hat{\mu} l_t) V^{-1} (w - \hat{\mu} l_t),$$

$$\hat{\mu} = [l_t V^{-1} l_t]^{-1} l_t V^{-1} w.$$
4. Application to the Polish stock market

This section examines evidence for fractional integration in Polish stock market. We investigate the long memory and persistence of the weekly returns of the trust fund Pioneer 1 from 1995 to 1998 (206 observations). We consider 18 different models for the first differences $\Delta y_t$ corresponding to all possible ARFIMA $(p, \delta, q)$ and ARMA $(p, q)$ for $p, q \leq 2$. All models have equal prior probabilities.

Table 1. Posterior model probabilities for ARFIMA $(p, \delta, q)$ and ARMA $(p, q)$ models

<table>
<thead>
<tr>
<th>$p$, $q$</th>
<th>ARFIMA $(p, \delta, q)$</th>
<th>ARMA $(p, q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0,0</td>
<td>0.1498</td>
<td>0.4734</td>
</tr>
<tr>
<td>0,1</td>
<td>0.0231</td>
<td>0.0610</td>
</tr>
<tr>
<td>0,2</td>
<td>0.0051</td>
<td>0.0219</td>
</tr>
<tr>
<td>1,0</td>
<td>0.0269</td>
<td>0.0750</td>
</tr>
<tr>
<td>1,1</td>
<td>0.0243</td>
<td>0.0659</td>
</tr>
<tr>
<td>1,2</td>
<td>0.0033</td>
<td>0.0129</td>
</tr>
<tr>
<td>2,0</td>
<td>0.0069</td>
<td>0.0237</td>
</tr>
<tr>
<td>2,1</td>
<td>0.0039</td>
<td>0.0108</td>
</tr>
<tr>
<td>2,2</td>
<td>0.0028</td>
<td>0.0093</td>
</tr>
<tr>
<td>Total</td>
<td>0.2461</td>
<td>0.7539</td>
</tr>
</tbody>
</table>

* Computed by the author.

The top two models are the ARMA $(0,0)$ with posterior probability 0.4734 and ARFIMA $(0, \delta, 0)$ with posterior probability 0.1498. In addition, all ARMA models receive posterior probability 0.7539 and ARFIMA models receive 0.2461. The posterior odds of unit root case versus fractional $\delta$ is three to one. Therefore in case of the weekly returns of the trust fund Pioneer 1 nonstationary models with unit root are more likely. The results confirm the efficient market hypothesis for the weekly returns. (posterior probability 0.47, while prior probability 0.056) However these facts do not negate the fractal behaviour of the returns. (posterior probability 0.15, while prior probability 0.056) It is much better results then for the other models – ARMA $(0, 1)$, $(1, 0)$ $(1, 1)$. Table 2 reports posterior means and standard deviation for $\delta$ for all the ARIFMA models.
Table 2. Posterior mean and standard deviation of $\delta$ for ARFIMA $(p, \delta, q)$ models.

<table>
<thead>
<tr>
<th>$p, q$</th>
<th>Mean</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0,0</td>
<td>0.0905</td>
<td>0.0587</td>
</tr>
<tr>
<td>0,1</td>
<td>0.1424</td>
<td>0.1072</td>
</tr>
<tr>
<td>0,2</td>
<td>0.0745</td>
<td>0.1162</td>
</tr>
<tr>
<td>1,0</td>
<td>0.0676</td>
<td>0.2456</td>
</tr>
<tr>
<td>1,1</td>
<td>0.0899</td>
<td>0.1842</td>
</tr>
<tr>
<td>1,2</td>
<td>0.0752</td>
<td>0.2099</td>
</tr>
<tr>
<td>2,0</td>
<td>-0.0529</td>
<td>0.2544</td>
</tr>
<tr>
<td>2,1</td>
<td>-0.0084</td>
<td>0.3054</td>
</tr>
<tr>
<td>2,2</td>
<td>0.0618</td>
<td>0.1983</td>
</tr>
</tbody>
</table>

* Computed by the author.

Figure 2 plots the posterior pdf of $\delta$ mixed over the 9 ARFIMA models.

Fig. 2. Posterior density of $\delta$

This plot indicates that the posterior pdf of $\delta$ is fairly Normal. The estimates presented in Table 2 are not very different from zero. The peak in the density at 0.0875 suggests some slight evidence of long memory. The regions of the parameter space which imply that $P(\delta > 0)$ receive 0.8889 posterior mass ($P(\delta > 0) = 0.8889$), while posterior probability that $\delta$ is less than zero is...
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0.1110. Therefore, there is strong evidence that \( I(\infty) = \infty \). Note however, that the posterior probability of \( \delta > 0 \) is downweighted by the data. (ARFIMA models receive 0.2461 posterior mass; Table 1).

The ARFIMA models have been criticised by Hauser, Pötcher, Reschnhofer (1992) for the purpose of estimating persistence. Finding the point estimates of \( \delta \) to be exactly equal to zero is very unlikely. Therefore, they argue that use of ARFIMA model corresponds to a prior information that the persistence measure is zero or \( \infty \). In their opinion ARFIMA models are not proper for the measurement of persistence. Following Koop, Ley, Osiewalski, Steel (1997) there are two responses to this criticism. First, apart from \( I(\infty) \) also ‘short-run’, ‘medium-run’ and ‘long-run’ persistence measure have been reported (Campbell and Mankiw (1987); Diebold, Rudebusch (1989)). Therefore, persistence can be estimated either for \( I(\infty) \) or for much shorter horizons (e.g. \( I(4), I(12), I(40) \)). Secondly, by putting some prior mass at \( \delta = 0 \), the posterior for \( I(\infty) \) contains two point masses at 0 and \( \infty \), but is continuous on \( (0, +\infty) \). However, no moments exists for such posterior (Koop, Osiewalski, Steel (1994)).

Table 3 reports impulse responses estimates for \( n = 4, 12, 40 \) for ARFIMA \((0,d,0)\), ARMA \((0,0)\), and for the ‘overall’ model which averages all 18 individual ARMA and ARFIMA models. Additionally, models which are a mixture of all 9 individual ARMA or ARFIMA models are also reported.

**Table 3.** Posterior means of impulse responses \( n = 4, 12, 40 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>ARFIMA ((0,d,0))</th>
<th>ARMA ((0,0))</th>
<th>Overall ARFIMA</th>
<th>Overall ARMA</th>
<th>Overall model</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1.206 (0.1399)</td>
<td>1.0000 (0.0000)</td>
<td>1.2196 (0.1484)</td>
<td>1.0659 (0.0846)</td>
<td>1.0103 (0.1483)</td>
</tr>
<tr>
<td>12</td>
<td>1.333 (0.2363)</td>
<td>1.0000 (0.0000)</td>
<td>1.3584 (0.2737)</td>
<td>1.0671 (0.1014)</td>
<td>1.0451 (0.2213)</td>
</tr>
<tr>
<td>40</td>
<td>1.504 (0.3769)</td>
<td>1.0000 (0.0000)</td>
<td>1.5501 (0.4893)</td>
<td>1.0895 (0.1074)</td>
<td>1.1092 (0.3381)</td>
</tr>
</tbody>
</table>

* Computed by the author.

For the random walk specification (ARMA \((0,0)\) for \( \Delta y_t \)) effect of any shock is exactly permanent and equal to one. Generally, all posterior means of impulse responses are very close to one for ARMA models. The results of applying the ARFIMA models are reported in column 2 and 4. The evidence from Table 3 confirm, that posterior probability \( P(\delta > 0) \) is more then 0.5. There-
fore, we can expect that for ARFIMA models posterior mean of impulse responses will tend to infinity as $n$ increases. Figure 3-6 plots the posterior density of impulse responses for $n = 4, 12, \text{ and } 40$.

Fig. 3. Posterior pdf of $I(4)$

Fig. 4. Posterior pdf of $I(12)$
The posteriors of impulse responses are similar across all compared models. Moreover, since the ARMA models have received \( \frac{3}{4} \) of the posterior mass, posterior densities of impulse responses for ARMA and overall models have a similar shape. The shape of these posteriors is unimodal with peaks at one. The posterior densities of impulse responses corresponding to the ARFIMA models are skewed with mean tending to infinity.

Figure 6 graphs impulse responses \( I(\infty) \) for ARMA and the overall model which is a mixture of 18 models. The impulse responses \( I(\infty) \) corresponding to the overall model has a point mass of 0.027 at zero and a point mass at infinity equal to 0.218. It also has continuous part on \( (0, +\infty) \).
5. Summary and conclusions

The ARFIMA models are natural extensions of ARMA representation. Allowing any $d > -1$ fractional integration provides parsimonious modeling of long memory processes. These processes have the ability to display significant dependence between two points in time as the distance between these points increases. Unlike short memory processes, their autocovariances decay monotonically and hyperbolically to zero as the lag increases. Following Koop, Ley, Osiewalski, Steel (1997) Bayesian analysis of ARFIMA models with implementation to measurement of persistence has been shown. Also in this paper long memory and persistence in the weekly returns of the trust fund Pioneer 1 on Polish stock market has been discussed. The results for ARFIMA $(0,\delta,0)$ with point estimate of 0.0875 suggest some slight evidence of long memory.
References


