Point-free geometry and topology
Part VI: Point-free geometry and verisimilitude of theories

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Outline

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Pseudo-metric spaces

Definition (of a pseudo-metric space)

A pseudo-metric space (pm-space for short) is a pair $\langle X, \delta \rangle$ such that $X \neq \emptyset$ and δ is a function from $X \times X$ to $[0, +\infty)$ with the following properties:

(M1) $\delta(x, x) = 0$

(M2) $\delta(x, y) = \delta(y, x)$

(M3) $\delta(x, y) \leqslant \delta(x, z) + \delta(z, y)$.

The function δ will be called pseudo-metric (p-metric), the number $\delta(x, y)$ pseudo-distance (p-distance) between x and y.

Examples of pm-spaces

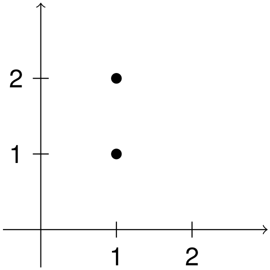
Example (The trivial pm-space)

A pair $\langle X, \delta \rangle$ such that for all $x, y \in X$, $\delta(x, y) = 0$.

Example (The absolute value pm-space)

A pair $\langle \mathbb{R}^2, \delta \rangle$ with $\delta \colon \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow [0, +\infty)$ such that:

$$\delta(\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle) := |x_1 - x_2| .$$



Approximate pseudo-metric spaces

Definition (of a point-free pseudometric space)

A point-free aproximate pseudometric space is a structure $\Re = \langle \mathbf{R}, \sqsubseteq, \delta, \Delta \rangle$ such that:

$$\langle \mathbf{R}, \sqsubseteq \rangle \text{ is a poset} \tag{APM0}$$

and $\delta \colon \mathbf{R} \times \mathbf{R} \longrightarrow [0, +\infty)$ and $\Delta \colon \mathbf{R} \longrightarrow [0, +\infty)$ are such that (set $\operatorname{diam}(x) \coloneqq \Delta(x, x)$):

$$\delta(x, x) = 0\,, \tag{APM1}$$

$$\delta(x, y) = \delta(y, x)\,, \tag{APM2}$$

$$\Delta(x, y) = \Delta(y, x) \tag{APM3}$$

$$0 \leqslant \Delta(x, y) - \delta(x, y) \leqslant \operatorname{diam}(x) + \operatorname{diam}(y)\,, \tag{APM4}$$

$$\delta(x, y) \leqslant \delta(x, z) + \delta(z, y) + \operatorname{diam}(z)\,, \tag{APM5}$$

$$\Delta(x, y) \leqslant \Delta(x, z) + \Delta(z, y)\,, \tag{APM6}$$

$$x_1 \sqsubseteq x_2 \wedge y_1 \sqsubseteq y_2 \Longrightarrow \delta(x_2, y_2) \leqslant \delta(x_1, y_1)\,, \tag{APM7}$$

$$x_1 \sqsubseteq x_2 \wedge y_1 \sqsubseteq y_2 \Longrightarrow \Delta(x_1, y_1) \leqslant \Delta(x_2, y_2)\,. \tag{APM8}$$

Approximate metric spaces

Definition (of an approximate metric space)

An apm-space $\mathfrak{R} = \langle \mathbf{R}, \sqsubseteq, \delta, \Delta \rangle$ is an approximate metric space (abbr: am-space) in case it satisfies the following condition:

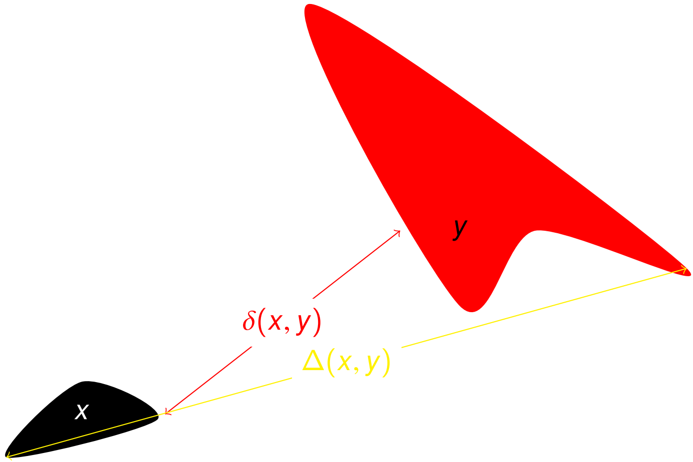
$$\Delta(x, y) = 0 \Longrightarrow x = y\,. \tag{AM}$$

Definition (of a point-region)

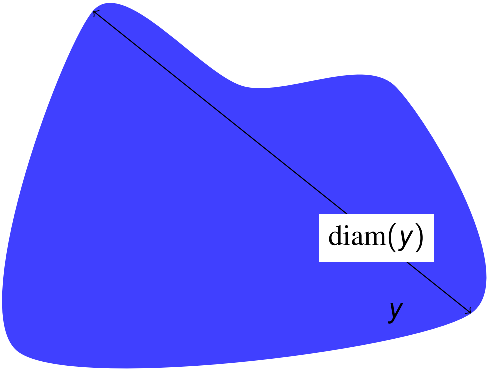
Let $\mathfrak{R} = \langle \mathbf{R}, \sqsubseteq, \delta, \Delta \rangle$ be an apm-space. A region $x \in \mathbf{R}$ is a point-region iff $\mathrm{diam}(x) = 0$. The set $\mathrm{PR}(\mathbf{R})$ is the set of all point-regions of $\mathfrak{R}$:

$$\mathrm{PR}(\mathbf{R}) := \{x \in \mathbf{R} \mid \mathrm{diam}(x) = 0\}\,. \tag{df PR(R)}$$

Distances between regions



Diameters of regions



Theories

Definition (of a theory)

A theory is a set of sentences.

Definition (of a true theory)

A theory is true iff all its sentences are true.

Definition (of a false theory)

A theory is false iff at least one of its sentences is false.

Consequences of theories

Definition (of a consequence of a theory)

A sentence p is a consequence of a theory T iff it follows from T (whenever theory T is true, then p must be true).

Fact

- *If a theory T is true, then all its consequences are true.*
- *If p is a consequence of T and p is false, then T is false as well.*

Closer to the truth

Definition

A theory T_2 is closer to the truth than a theory T_1 iff

- all true consequences of T_1 are also consequences of T_2
- all false consequences of T_2 are also consequences of T_1
- either there is some true consequence of T_2 which is not a consequence of T_1 or there is some false consequence of T_1 which is not a consequence of T_2.

First-order theories

- Let $\mathcal{L}$ be any first-order language.
- Let $\mathrm{For}(\mathcal{L})$ be the set of all formulas of $\mathcal{L}$.

Definition (of a first-order theory)

A first-order theory built in language $\mathcal{L}$ is a triple:

$$\mathrm{T} := \langle \mathcal{L}, \mathrm{X}, \vdash \rangle$$

such that

- $\mathrm{X} \subseteq \mathrm{For}(\mathcal{L})$, it will be called the set of specific axioms of T
- $\vdash$ is a provability (derivability) relation.

Remark

For a fixed theory T the set of its specific axioms will be denoted by '$\mathrm{Axm}(\mathrm{T})$'.

Some special sets of formulas

Definition

Let $\mathrm{X} \subseteq \mathrm{For}(\mathcal{L})$ for some first-order language $\mathcal{L}$. We define:

- the set of formulas provable from X:

$$\mathrm{Prov}(\mathrm{X}) := \{F \in \mathrm{For}(\mathcal{L}) \mid \mathrm{X} \vdash F\}.$$

- the set of formulas refutable from X:

$$\mathrm{Ref}(\mathrm{X}) := \{F \in \mathrm{For}(\mathcal{L}) \mid \mathrm{X} \vdash \neg F\}.$$

- the set of formulas which are decidable on base of X:

$$\mathrm{Dec}(\mathrm{X}) := \mathrm{Prov}(\mathrm{X}) \cup \mathrm{Ref}(\mathrm{X}).$$

- the elements of $\complement\mathrm{Dec}(\mathrm{X})$ will be called undecidable formulas given X.

Consistent sets of formulas

Definition (of a consistent set of formulas)

A set $\mathrm{X} \subseteq \mathrm{For}(\mathcal{L})$ is consistent iff no formula is both provable and refutable from X:

$$\mathrm{cons}(\mathrm{X}) \overset{\mathrm{df}}{\iff} \mathrm{Prov}(\mathrm{X}) \cap \mathrm{Ref}(\mathrm{X}) = \emptyset\,. \qquad (\mathsf{df}\,\mathrm{cons}(X))$$

Binary operations on sets of formulas

Definition

Let $X, Y \subseteq \mathrm{For}(\mathcal{L})$ for some first-order language $\mathcal{L}$. We define:

- the formulas on which X and Y agree:

$$\mathrm{Agree}(X, Y) := \big(\mathrm{Prov}(X) \cap \mathrm{Prov}(Y)\big) \cup \big(\mathrm{Ref}(X) \cap \mathrm{Ref}(Y)\big).$$

- the formulas on which X and Y disagree:

$$\mathrm{Disagree}(X, Y) := \big(\mathrm{Prov}(X) \cap \mathrm{Ref}(Y)\big) \cup \big(\mathrm{Prov}(Y) \cap \mathrm{Ref}(X)\big).$$

We also set the following terminology:

- $\complement \mathrm{Agree}(X, Y)$ will be called the set on which X and Y do not agree or may disagree,
- $\complement \mathrm{Disagree}(X, Y)$ will be called the set on which X and Y do not disagree or may agree.

Some important facts

If X and Y are both consistent, then:

$$\mathrm{Agree}(\mathrm{X},\mathrm{Y}) \cap \mathrm{Disagree}(\mathrm{X},\mathrm{Y}) = \emptyset\,, \tag{1}$$

$$\mathrm{Agree}(\mathrm{X},\mathrm{Y}) \subseteq \complement\mathrm{Disagree}(\mathrm{X},\mathrm{Y})\,, \tag{2}$$

$$\mathrm{Disagree}(\mathrm{X},\mathrm{Y}) \subseteq \complement\mathrm{Agree}(\mathrm{X},\mathrm{Y})\,, \tag{3}$$

$$\mathrm{Dec}(\mathrm{X}) \cap \mathrm{Dec}(\mathrm{Y}) = \mathrm{Agree}(\mathrm{X},\mathrm{Y}) \cup \mathrm{Disagree}(\mathrm{X},\mathrm{Y})\,, \tag{4}$$

$$\mathrm{Disagree}(\mathrm{X},\mathrm{Y}) = \complement\mathrm{Agree}(\mathrm{X},\mathrm{Y}) \cap \mathrm{Dec}(\mathrm{X}) \cap \mathrm{Dec}(\mathrm{Y})\,, \tag{5}$$

$$\mathrm{Disagree}(\mathrm{X},\mathrm{Y}) \subseteq \mathrm{Dec}(\mathrm{X}) \cap \mathrm{Dec}(\mathrm{Y})\,, \tag{6}$$

$$\mathrm{Disagree}(\mathrm{X},\mathrm{Y}) \cap \left(\complement\mathrm{Dec}(\mathrm{X}) \cup \complement\mathrm{Dec}(\mathrm{Y})\right) = \emptyset\,. \tag{7}$$

Tests and measures

Definition (of a sentence)

$F \in \mathrm{For}(\mathcal{L})$ is a sentence iff F has no free variables. Let $\mathrm{Sent}(\mathcal{L})$ be the set of all sentences built in $\mathcal{L}$.

Definition (of a test)

Let $F \in \mathrm{Sent}(\mathcal{L})$. By a test we mean the set of all formulas which are equivalent either to F or to its negation:

$$\|F\| := \left\{G \in \mathrm{Sent}(\mathcal{L}) \mid \mathrm{T} \vdash G \equiv F \text{ or } \mathrm{T} \vdash G \equiv \neg F\right\}. \qquad (\mathrm{df}\,\|\ \|)$$

For a set $\mathrm{X} \subseteq \mathrm{Sent}(\mathcal{L})$ let:

$$\|\mathrm{X}\| := \left\{\|F\| \mid F \in \mathrm{X}\right\}.$$

Tests and measures

Definition (of a vacuous test)

Let $\perp$ be an arbitrary (but fixed) logically false formula. Bya vacuous test we mean the set $\|\perp\|$:

$$\mathbf{0} := \|\perp\| \,. \tag{df $\mathbf{0}$}$$

Definition

By a relevance measure we mean a function

$$\mathrm{rel}\colon\ \|\mathrm{Sent}(\mathcal{L})\| \longrightarrow [0,1]$$

such that:

$$\begin{aligned} \mathrm{rel}(\mathbf{0}) &:= 0\,, \\ \sum_{x\in\|\mathrm{Sent}(\mathcal{L})\|} \mathrm{rel}(x) &:= 1\,. \end{aligned} \tag{df rel}$$

Tests and measures

Definition (of a relevant test, relevant sentence and sensitive function)

- $X \in \|\mathrm{Sent}(\mathcal{L})\|$ is relevant iff $\mathrm{rel}(X) \neq 0$.
- A sentence F is relevant iff $\|F\|$ is relevant.
- A function rel is sensitive iff all non-vacuous tests are relevant.

Definition (of a measure of the degree of relevance)

By a measure of the degree of relevance of $\mathrm{X} \subseteq \|\mathrm{Sent}(\mathcal{L})\|$ (with respect to some rel function) we mean the function:

$$\mu \colon \mathcal{P}(\|\mathrm{Sent}(\mathcal{L})\|) \longrightarrow [0, 1]$$

such that:

$$\mu(X) := \sum_{x \in \mathrm{X}} \mathrm{rel}(x)\,. \qquad (\mathrm{df}\,\mu)$$

Distances between sets of sentences

Definition (of distances between sets of sentences)

Let rel be some relevance measure.

- Let
$$d\colon \mathcal{P}(\mathrm{Sent}(\mathcal{L})) \times \mathcal{P}(\mathrm{Sent}(\mathcal{L})) \longrightarrow [0,1]$$
be such that:
$$d(\mathrm{X}_1, \mathrm{X}_2) := \mu\Big(\|\mathrm{Disagree}(\mathrm{X}_1, \mathrm{X}_2)\|\Big). \tag{8}$$

- On the other hand, let
$$D\colon \mathcal{P}(\mathrm{Sent}(\mathcal{L})) \times \mathcal{P}(\mathrm{Sent}(\mathcal{L})) \longrightarrow [0,1]$$
be such that:
$$D(\mathrm{X}_1, \mathrm{X}_2) := \mu\Big(\|\complement\mathrm{Agree}(\mathrm{X}_1, \mathrm{X}_2)\|\Big). \tag{9}$$

Distances between sets of sentences

Definition (of the actual and possible contrasts)

For given $\mathrm{X}_1, \mathrm{X}_2 \subseteq \mathrm{Sent}(\mathcal{L})$:

- the number $d(\mathrm{X}_1, \mathrm{X}_2)$ is called the actual contrast between the two sets in question,
- the number $D(\mathrm{X}_1, \mathrm{X}_2)$ is called the possible contrast between them.

Distances between sets of sentences

Definition (of the diameter of the set of sentences)

Let $\mathrm{X} \subseteq \mathrm{Sent}(\mathcal{L})$. According to the earlier definitions of the diameter of a given set and of the possible contrast between sets of sentences we have:

$$\mathrm{diam}(\mathrm{X}) := D(\mathrm{X},\mathrm{X}) := \mu(\|\complement\mathrm{Agree}(\mathrm{X},\mathrm{X})\|) = \mu(\|\complement\mathrm{Dec}(\mathrm{X})\|)\,.$$

Fact

From the definition of μ it follows that:

$$\mathrm{diam}(\mathrm{X}) = 1 - \mu(\|\mathrm{Dec}(\mathrm{X})\|)\,. \tag{10}$$

Spaces of sets sentences

Definition

Let **CA** be the set of all consistent sets of sentences built in first-order language $\mathcal{L}$:

$$\mathbf{CA} := \{X \in \mathrm{Sent}(\mathcal{L}) \mid \mathrm{cons}(X)\}\,. \qquad (\mathrm{df}\ \mathbf{CA})$$

Definition

Let $\langle \mathbf{CA}, \sqsubseteq \rangle$ be such that:

$$X \sqsubseteq Y \overset{\mathrm{df}}{\iff} Y \subseteq X\,.$$

Fact

The structure $\langle \mathbf{CA}, \sqsubseteq, d, D \rangle$ is an apm-space.

Spaces of theories

Fact

If rel *is sensitive, then:*

$$\mu(\|X\|) = 1 \iff \mathrm{Rel}(\mathcal{L}) \subseteq X\,.$$

Theorem

The following equivalences hold in $\langle \mathbf{CA}, \sqsubseteq, d, D \rangle$*:*

1. $d(\mathrm{X}, \mathrm{Y}) = 1 \iff \mathrm{Disagree}(\mathrm{X}, \mathrm{Y}) = \mathrm{Rel}(\mathcal{L})$,
2. $d(\mathrm{X}, \mathrm{Y}) = 0 \iff \mathrm{Disagree}(\mathrm{X}, \mathrm{Y}) = \emptyset$,
3. $D(\mathrm{X}, \mathrm{Y}) = 1 \iff \mathrm{Agree}(\mathrm{X}, \mathrm{Y}) \cap \mathrm{Rel}(\mathcal{L}) = \emptyset$,
4. $D(\mathrm{X}, \mathrm{Y}) = 0 \iff \mathrm{Agree}(\mathrm{X}, \mathrm{Y}) = \mathrm{Sent}(\mathcal{L})$,
5. $\mathrm{diam}(\mathrm{X}) = 1 \iff \mathrm{Dec}(\mathrm{X}) \cap \mathrm{Rel}(\mathcal{L}) = \emptyset$,
6. $\mathrm{diam}(\mathrm{X}) = 0 \iff \mathrm{Dec}(\mathrm{X}) = \mathrm{Sent}(\mathcal{L})$.

Spaces of theories

Definition (of the set of theorems)

Suppose T is a first-order theory. The set of theorems of T is the set of all formulas which are provable from the specific axioms of T:

$$\mathrm{Th}(\mathrm{T}) := \{F \in \mathrm{For}(\mathcal{L}) \mid \mathrm{Axm}(\mathrm{T}) \vdash F\}. \qquad (\text{df } \mathrm{Th}(\mathrm{T}))$$

That is $\mathrm{Th}(\mathrm{T}) = \mathrm{Prov}(\mathrm{Axm}(\mathrm{T}))$.

Definition (of the set of deductive closures)

Let **Th** be the set of all closures of sets of sentences built in a first-order language $\mathcal{L}$:

$$\mathbf{Th} := \{\mathrm{Th}(\mathrm{T}) \mid \mathrm{T} \text{ is a first-order theory}\}. \qquad (\text{df } \mathbf{Th})$$

Of course, **Th** $= \{\mathrm{Prov}(\mathrm{X}) \mid \mathrm{X} \subseteq \mathrm{Sent}(\mathcal{L})\}$.

Remark

From now on by a theory we mean any element of **Th**, that is any set of theorems of some first-order theory built in a language $\mathcal{L}$.

Spaces of theories

Definition (of the apm-space of all theories)

The subspace $\langle \mathbf{Th}, \sqsubseteq, d, D\rangle$ of the space $\langle \mathbf{CA}, \sqsubseteq, d, D\rangle$ will be called the apm-space of theories.

Definition (of the apm-space of finitely axiomatizable theories)

Let **Fin** be the set of all finitely axiomatizable theories. The subspace $\langle \mathbf{Fin}, \sqsubseteq, d, D\rangle$ of the space $\langle \mathbf{Th}, \sqsubseteq, d, D\rangle$ is the apm-space of all finitely axiomatizable theories.

Definition (of the apm-space of complete theories)

Let **Comp** be the class of all complete theories. The subspace $\langle \mathbf{Comp}, \sqsubseteq, d, D\rangle$ of the space $\langle \mathbf{Th}, \sqsubseteq, d, D\rangle$ is the apm-space of all complete theories.

Spaces of theories

Fact

In $\langle \mathbf{Comp}, \sqsubseteq, d, D\rangle$ it is the case that:

- $d(\mathrm{T}_1, \mathrm{T}_2) = D(\mathrm{T}_1, \mathrm{T}_2)$,
- $\mathrm{T}_1 \sqsubseteq \mathrm{T}_2$ *if and only if* $\mathrm{T}_1 = \mathrm{T}_2$,
- **Comp** *is a metric space.*

Spaces of theories

Theorem

In the space of all theories $\langle \mathbf{Th}, \sqsubseteq, d, D \rangle$ *for every* $\mathrm{T} \in \mathrm{Th}$*:*

$$\mathrm{T} = \bigcup \mathrm{PR}(\mathrm{T})\,.$$

Moreover:

$$\begin{aligned} d(\mathrm{T}_1, \mathrm{T}_2) = 0 &\iff \mathrm{T}_1 \bigcirc \mathrm{T}_2 \\ &\iff \exists_{\mathrm{T} \in \mathrm{PR}(\mathbf{Th})} \mathrm{T} \in \mathrm{T}_1 \cap \mathrm{T}_2 \\ &\iff \mathrm{cons}(\mathrm{T}_1 \uplus \mathrm{T}_2)\,. \end{aligned}$$

Spaces of theories

Definition (of tangent theories)

Theories T_1 and T_2 are tangent iff

$$|\mathrm{T}_1 \cap \mathrm{T}_2 \cap \mathrm{PR}(\mathbf{Th})| = 1 \,.$$

Theorem

Theories T_1 *and* T_2 *are tangent iff* $\mathrm{cons}(\mathrm{T}_1 \uplus \mathrm{T}_2)$ *and* $\mathrm{T}_1 \uplus \mathrm{T}_2 \in \mathbf{Comp}$.

Spaces of theories

Definition (of an essentially incomplete theory)

A theory T is essentially incomplete iff it is incomplete and every its finite extension is incomplete.

Theorem

The following statements are equivalent:

- T *is essentially incomplete,*
- T *is tangent to no element of* **Fin***,*
- $\mathrm{PR}(\mathbf{Th}) \cap \mathrm{T} \neq \emptyset$ *and* $\mathrm{PR}(\mathbf{Fin}) \cap \mathrm{T} = \emptyset$*.*

True sentences

Definition (of true sentences of a given theory)

- Suppose that T is a first-order theory which was built to axiomatize some distinguished (standard) structure $\mathfrak{M}$.
- By the set of all true sentences we will mean the of all these sentences of $\mathcal{L}$ which are satisfied by $\mathfrak{M}$:

$$\mathbf{V} := \{F \in \mathrm{Sent}(\mathcal{L}) \mid \mathfrak{M} \models F\}\,. \tag{11}$$

Verisimilitude as an approximate distance

Fact

For any theory T *completeness of* $\mathbf{V}$ *entails:*

$$D(\mathrm{T}, \mathbf{V}) = d(\mathrm{T}, \mathbf{V}) + \mathrm{diam}(\mathrm{T})\,.$$

Verisimilitude as an approximate distance

- Verisimilitude is to reflect the distance of a theory from truth.
- Thus, by the previous fact, it hinges upon $d(\mathrm{T}, \mathbf{V})$ and $\mathrm{diam}(\mathrm{T})$
- Verisimilitude attains maximal value (say 1) when $\mathrm{T} = \mathbf{V}$, that is in case $d(\mathrm{T}, \mathbf{V}) = 0$ and $\mathrm{diam}(T) = 0$.

Definition (of a verisimilitude measure)

A verisimilitude measure is a function $\mathrm{Vs}\colon \mathbf{Th} \longrightarrow [0, 1]$ such that:

$$\mathrm{Vs}(\mathrm{T}) := 1 - \Big(t \cdot d(\mathrm{T}, \mathbf{V}) + i \cdot \mathrm{diam}(\mathrm{T})\Big). \qquad (\mathrm{df\ Vs})$$

where $0 < i \leqslant t \leqslant 1$ correspond to the weight assigned to the closeness of T from truth and the degree of incompleteness of the theory.

The End
of
Part VI