Point-free geometry and topology
Part II: Mereology

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Outline

Two notions of *set*

Parts

Basic axioms

Mereological sum

Classical mereology

Some theorems of classical mereology

The first notion of *set*

According to the first notion a set is an object that consist of parts. It is a fusion of a conglomerate of objects. For example:

- a brick wall can be seen as a conglomerate that consist of bricks and concrete,
- the territory of the United States of America can be seen as vast physical object which is a fusion of territories of its states.

In both cases we can say that a brick is an element of the wall, and that California is an element of the United States.

Sets as described above will be called by us fusions or mereological sets.

Elements are parts

It is not hard to notice however that we use the notion of *element* in a very peculiar sense, that is a brick is part of the wall and California is part of the U.S.A.

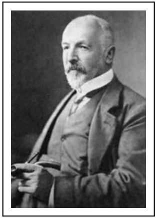
Thus we use the term ‘is an element of’ in the sense of being a part of some object. In this sense:

- my hand is an element of me,
- the Moon is an element of the Solar System,
- Capri island is an element of Italy,
- Italy is an element of Europe.

The second notion of *set*

The second notion of *set* is common for contemporary mathematics. Let me remind that according to Georg Cantor:

a set is collection into a whole of definite distinct objects of our intuition or of our thought.



Two ways of creating sets in mathematics

The second notion of *set*

In mathematics we can create sets in two basic ways[1]:

- extensional notion of a set: in this case we have many objects, all of those objects or some of them can be collected together to form a set, and this process can be repeated *ad infinitum*;
- intensional notion of a set: the set is understood as the extension of a concept or property, in a sense that it contains as its elements all objects that have this property.

[1] Hao Wang *What is logic?*

Basic property for sets

Some properties of Cantorian sets

Sets as understood by mathematics are often called distributive ones, I will also often call them Cantorian ones.

Cantorian sets satisfy the following basic property. Suppose that letter 'S' represents a place in which we can put a name. Then we can say that:

$$\forall_x (x \text{ is an element of the set of } S\text{-es} \iff x \text{ is } S)\,, \tag{B}$$

or using standard set theoretical notation:

$$\forall_x (x \in \{y \mid y \text{ is } S\} \iff x \text{ is } S)\,. \tag{B}$$

Sets are abstract objects

Some properties of Cantorian sets

Using (B) we can argue after Quine[2] that Cantorian sets must be abstract objects, that is they do not occupy space-time.

Proof.

- Consider a heap of stones.
- According to (B):

$$\text{the set of stones} \neq \text{the set of molecules.}$$

- Suppose they are concrete objects.
- Thus:

  the set of stones $=$ the heap of stones $=$ the set of molecules.
- In consequence the set of stones $=$ the set of molecules.

□

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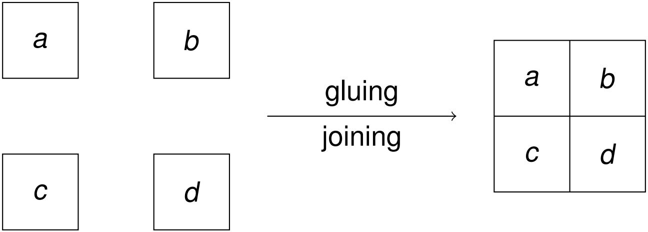
[2] *Logic and the reification of universals* [in:] From a logical point of view

(B) does not hold for fusions

Comparing two notions

If we have some objects, then the process of gathering them to form a fusion may be roughly compared to the act of «gluing» these objects into a whole.



(B) does not hold for fusions

Comparing two notions

And this is different process from that of forming a Cantorian set.

$$\begin{array}{cc} \boxed{a} & \boxed{b} \\ \boxed{c} & \boxed{d} \end{array} \xrightarrow{\text{collecting}} \{a, b, c, d\}$$

(B) does not hold for fusion

Comparing two notions

Let me introduce some notation now. If we have some objects $a_1, \ldots, a_n$ by means of:

$$[\![a_1, \ldots, a_n]\!]$$

I will denote the object that is the result of «gluing» or «joining» this objects. Similarly, in case $\varphi(x)$ is a condition put upon x (where x is a free variable), by

$$[\![x \mid \varphi(x)]\!]$$

I will denote the object that is the result of joining all x-es satisfying φ. Thus, for example:

$$[\![x \mid x \text{ is an American state}]\!] = \text{U.S.A.}$$

(B) does not hold for fusions

Comparing two notions

From the explanations above we can see that there is a substantial difference in forming fusions and Cantorian sets.

- elements of elements of a, b, c, d (if there are any) are elements of $[\![a, b, c, d]\!]$,
- elements of elements of a, b, c, d (if there are any) do not have to be elements of $\{a, b, c, d\}$.

I of course use the term ‘element’ in two different meanings.

(B) does not hold for fusions

Comparing two notions

Fusions do not satisfy the principle (B). To explain this, suppose that we are considering some group of S-es and their fusion $[\![x \mid x \text{ is } S]\!]$. It is fairly obvious that:

$$\forall_x (x \text{ is } S \Longrightarrow x \text{ is an element of } [\![x \mid x \text{ is } S]\!])\,. \tag{1}$$

However the converse implication is not generally true, that is:

$$\neg\forall_x (x \text{ is an element of } [\![x \mid x \text{ is } S]\!] \Longrightarrow x \text{ is } S)\,. \tag{2}$$

For example:

$$\begin{aligned} &\text{Yellowstone is an element of } [\![x \mid x \text{ is an American state}]\!] \\ &\qquad \wedge \text{Yellowstone is not a state}\,. \end{aligned} \tag{3}$$

As it could be seen from the characterizations given above the term ‘is an element of’ was used meaning is part of. Theory of parthood is usually called mereology from Greek *meros* which means *part*.

- The creator of mereology: polish logician and mathematician Stanisław Leśniewski (1886-1939)
- mereology can be characterized as a theory of *fusions* in opposition to the Cantorian notion of *set*

What was Leśniewski's motivations for developing mereology?

- Leśniewski on Cantorian sets: *I can feel in them smell of mythical objects from rich gallery of figments of the imagination*.
- nothing like the empty set can exists
- his ontological stance admitted only concrete (spatio-temporal) objects
- series of papers titled *On foundations of mathematics*.

As it was said Leśniewski did not recognize existence of abstract objects, like Cantorian sets for example. From ontological point of view mereology as theory of fusions is better for nominalism. The main reasons for this are:

- first, in the process of joining objects to form fusions ontological status of fusion may be inherited from that of its constituents, thus if we fuse concrete objects what we obtain may be a concrete object; this is different from Cantorian sets which are always abstract entities;
- second, from nominalistic point of view it is natural to talk about parts of objects, while set theoretical $\in$ has nothing to do with any relationship between concrete objects.

The name 'calculus of individuals' is used as well to underline nominalistic foundations of the theory.

Contemporary mereology is far from Leśniewski's intentions.

- It is usually done by means of set theoretical tools.
- It is used to build theories of abstract objects (point-free geometry, point-free topology).

Basic properties of parthood

Mereology as a theory of part of relation

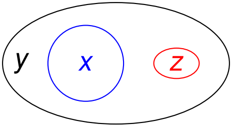
- Asymmetry: if x is part of y, then y is not part of x.
- Irreflexivity: nothing is part of itself.
- Transitivity: if x is part of y and y part of z, then x is part of z.

Weak supplementation principle

Mereology as a theory of part of relation

Weak Supplementation Principle

If *x* is part of *y*, then there must be some *z* which is part of *x* but is exterior to *y*.



Strong Supplementation Principle

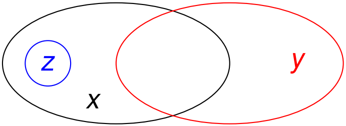
Mereology as a theory of part of relation

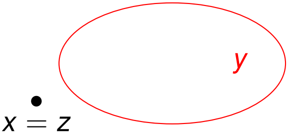
Definition

A mereological atom is an object that has no parts.

Strong Supplementation Principle

If x is not part of y neither it is identical with y, then there must be some z which either is part of x or is identical with x but is exterior to y.





Short summary – basic properties of parthood

Mereology as a theory of part of relation

- irreflexivity
- transitivity
- asymmetry
- weak supplementation principle
- strong supplementation principle

Basic notation

- By means of letter 'M' I will denote the domain, letters 'a', 'b', 'c', 'd', 'u', 'v', 'w', 'x', 'y', 'z' will be used to denote elements of M that I will call, in a standard way, *objects*.
- I will use symbol '$\sqsubset$' to denote *parthood*, thus '$x \sqsubset y$' is read: x is part of y. So $\sqsubset$ is a binary relation in M: $\sqsubset \subseteq M \times M$:

$$\sqsubset := \{\langle x, y\rangle \mid x \text{ is part of } y\}\,.$$

- '$x \not\sqsubset y$' is to mean $\neg x \sqsubset y$.

Asymmetry and transitivity

Basic axioms

$$\forall_{x \in M} \forall_{y \in M} (x \sqsubset y \implies y \not\sqsubset x) \,, \tag{A}$$

$$\forall_{x \in M} \forall_{y \in M} \forall_{z \in M} (x \sqsubset y \land y \sqsubset z \implies x \sqsubset z) \,. \tag{T}$$

Fact

$\forall_{x \in M} x \not\sqsubset x$

Proof.

Suppose that there is some $x \in M$ such that $x \sqsubset x$. Then by (A) we have that $x \not\sqsubset x$, a contradiction. □

Asymmetry and transitivity

Basic axioms

Neither Weak Supplementation Principle nor Strong Supplementation Principle follow from (A) and (T).

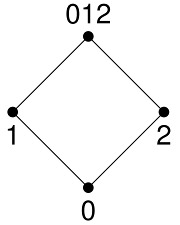
Model

Weak Supplementation Principle does not follow from (A) and (T).

$0 \sqsubset 1$ but there is no object which is part of 1 and exterior to 0

Model

Strong Supplementation Principle does not follow from (A) and (T).



1 is not part nor identical with 2 but there is no object which is part of or identical with 2 but exterior to 1

Definition

Object x is an ingrediens of object y if and only if x is part of y or x is identical with y.

$$x \sqsubseteq y \overset{\text{df}}{\iff} x \sqsubset y \vee x = y\,. \tag{df $\sqsubseteq$}$$

$\boxed{\text{Basic facts about ingrediens relation}}$

$$\forall_{x \in M} x \sqsubseteq x\,, \tag{4}$$

$$\forall_{x \in M} \forall_{y \in M} (x \sqsubseteq y \wedge y \sqsubseteq x \Longrightarrow x = y)\,, \tag{5}$$

$$\forall_{x \in M} \forall_{y \in M} \forall_{z \in M} (x \sqsubseteq y \wedge y \sqsubseteq z \Longrightarrow x = z)\,, \tag{$\mathrm{T}_{\sqsubseteq}$}$$

$$\forall_{x \in M} \forall_{y \in M} (x \sqsubseteq y \Longrightarrow \forall_{z \in M} (z \sqsubseteq x \Longrightarrow z \sqsubseteq y))\,. \tag{6}$$

Auxiliary relations

Basic axioms

Extensionality for ingrediens relation

If objects have exactly the same ingredienses, then they are identical:

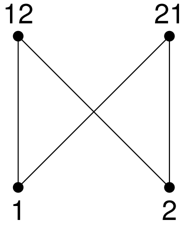
$$\forall_{x\in M}\forall_{y\in M}(\forall_{z\in M}(z\sqsubseteq x \iff z\sqsubseteq y)\implies x=y)\,.$$

Proof.

- This follows from reflexivity and antisymmetry of $\sqsubseteq$.
- Take arbitrary x and y and assume that $\forall_{z\in M}(z\sqsubseteq x \iff z\sqsubseteq y)$.
- Since $x\sqsubseteq x$, we have that $x\sqsubseteq y$.
- Similarly, since $y\sqsubseteq y$ it is the case that $y\sqsubseteq x$.
- Thus by antisymmetry we have that $x=y$.

$\square$

Notice that from axioms (A) and (T) does not follow that if objects have the same parts, then they are identical. The following model show that this is not the case.



Objects 12 and 21 have exactly the same parts (objects 1 and 2) but $12 \neq 21$. They have different ingredienses of course.

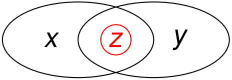
Definition

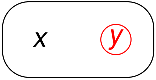
Object x overlaps object y if and only if there is some z which is ingrediens of both x and y.

$$x \bigcirc y \overset{\text{df}}{\iff} \exists_{z \in M}(z \sqsubseteq x \land z \sqsubseteq y)\,. \tag{df $\bigcirc$}$$

Example

Two examples of overlapping objects.





Auxiliary relations

Basic axioms

Basic facts about overlapping

$$\forall_{x \in M} x \bigcirc x \,, \tag{7}$$

$$\forall_{x \in M} \forall_{y \in M} (x \bigcirc y \implies y \bigcirc x) \,, \tag{8}$$

$$\forall_{x \in M} \forall_{y \in M} (x \sqsubseteq y \lor y \sqsubseteq x \implies x \bigcirc y) \,, \tag{9}$$

$$\forall_{x \in M} \forall_{y \in M} \forall_{z \in M} (x \sqsubseteq y \land z \bigcirc x \implies z \bigcirc y) \,. \tag{10}$$

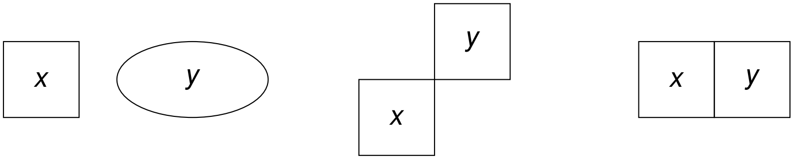
Definition

Object x is exterior to object y iff they do not overlap, that is there is no z which is ingrediens of x and y:

$$\begin{aligned} x \wr y &\overset{\text{df}}{\iff} \neg x \bigcirc y && \text{(df}\wr\text{)} \\ &\iff \neg\exists_{z\in M}(z \sqsubseteq x \wedge z \sqsubseteq y)\,. \end{aligned}$$

Example

A couple of situations in which objects are exterior to each other.



Auxiliary relations

Basic axioms

Basic facts about being exterior to

$$\neg\exists_{x\in M} x \wr x\,, \tag{11}$$

$$\forall_{x\in M}\forall_{y\in M}(x \wr y \Longrightarrow y \wr x)\,, \tag{12}$$

$$\forall_{x\in M}\forall_{y\in M}(x \wr y \Longrightarrow x \not\sqsubseteq y \wedge y \not\sqsubseteq x)\,, \tag{13}$$

$$\forall_{x\in M}\forall_{y\in M}\forall_{z\in M}(x \sqsubseteq y \wedge y \wr z \Longrightarrow x \wr z)\,. \tag{14}$$

Strong Supplementation Principle formally

Basic axioms

We take Strong Supplementation Principle to be <u>the third axiom</u>.

$$\forall_{x\in M}\forall_{y\in M}(x \not\sqsubseteq y \Longrightarrow \exists_{z\in M}(z \sqsubseteq x \wedge z \wr y))\,. \qquad \text{(SSP)}$$

In the theory of partially ordered sets (SSP) is usually called *separation condition*.

What about Weak Supplementation Principle?

- First, let me express it in a formal way.

$$\forall_{x \in M} \forall_{y \in M} (x \sqsubset y \Longrightarrow \exists_{z \in M} (z \sqsubset y \wedge z \wr y)) . \qquad \text{(WSP)}$$

- Second, we DO NOT have to take it as another axiom.

SSP vs. WSP

Basic axioms

Theorem

(WSP) is a consequence of axioms (A) and (SSP) and definitions (df $\sqsubseteq$), (df $\bigcirc$) and (df $\wr$).

Proof.

- Assume that $x \sqsubset y$.
- This means that: (a) $x \neq y$, (b) $x \bigcirc y$, (c) $y \not\sqsubseteq x$.
- By (c) and (SSP) there is z such that (d) $z \sqsubseteq y$ and (e) $z \wr x$.
- But $z \neq y$ (otherwise it would have to be the case that $x \bigcirc z$).
- So $z \sqsubset y$ and $z \wr x$.

□

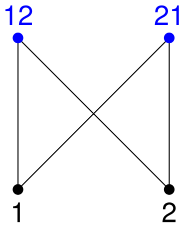
Where did we use assymetry?

SSP vs. WSP

Basic axioms

Fact

(SSP) *does not follow from* (A), (T) *and* (WSP).



- (A), (T) and (WSP) hold in this case.
- However (SSP) fails.
- Take objects 12 and 21.
- It is the case that $12 \not\sqsubseteq 21$.
- However $\neg\exists_{z\in M}(z \sqsubseteq 12 \wedge z \wr 21)$ since everything overlaps 21.

The smallest element usually does not exist

Basic axioms

$$\mathrm{Card}\, M > 1 \iff \exists_{x,y \in M} x \wr y\,.$$

Proof.

- Let $\mathrm{Card}\, M > 1$.
- So there are x_1 and x_2 such that (a) $x_1 \neq x_2$.
- Either $x_1 \wr x_2$ or $x_1 \bigcirc x_2$.
- There is $z \in M$ such that $z \sqsubseteq x_1$ and $z \sqsubseteq x_2$.
- By (a), either $z \sqsubset x_1$ or $z \sqsubset x_2$.
- Thus, by (WSP), there is z_0 such that either $z_0 \sqsubset x_1$ and $z_0 \wr z$ or $z_0 \sqsubset x_2$ and $z_0 \wr z$.

□

The smallest element usually does not exist

Basic axioms

Corrolary

The domain M contains the smallest element (with respect to $\sqsubseteq$) iff M has only one element:

$$\exists_{x \in M} \forall_{y \in M} x \sqsubseteq y \iff \operatorname{Card} M = 1 \,. \tag{$\nexists$0}$$

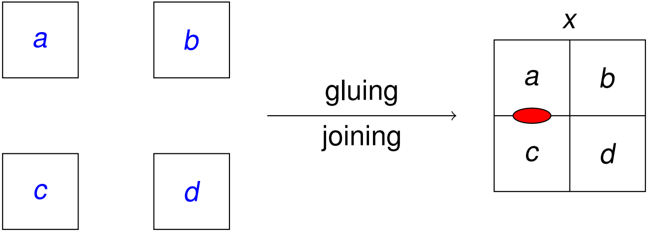
Where are we now?

- We have adopted three axioms: (A), (T) and (SSP).
- Fundamental consequences of these are:
  - irreflexivity of parthood,
  - (WSP).
  - Non-existence of the smallest element.

Mereological sum

- We are now going to formalize the notion of mereological sum or fusion.
- Recall that what we are aiming at with these notions is to model the process of joining objects into one single entity.

Mereological sum – definition



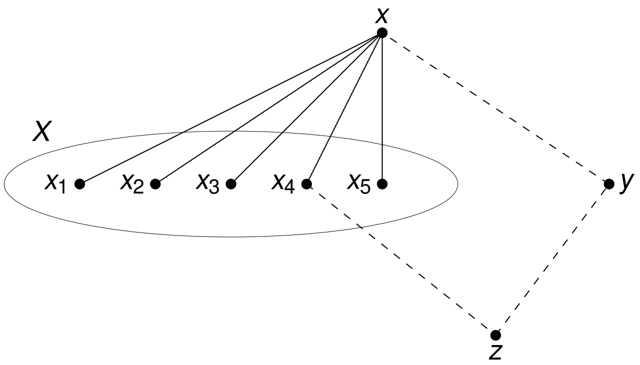
- Every element a, b, c and d is part of x (the more so is ingrediens of x).
- Whatever part (or ingrediens) of x we take, then it must overlap at least one of objects a, b, c or d.

Definition (Mereological sum – informally)

An object x is a mereological sum of all elements of Z iff every element of Z is an ingrediens of x and every ingrediens of x overlaps some element z from Z

Mereological sum

Definition



Mereological sum – definition

Definition (Mereological sum – formally)

We define a «hybrid» relation **Sum** $\subseteq M \times \mathcal{P}(M)$:

$$x\ \mathbf{Sum}\ Z \overset{\text{df}}{\iff} \forall_{z \in Z}\, z \sqsubseteq x \wedge \forall_{y \in M}(y \sqsubseteq x \implies \exists_{z \in M}\, y \bigcirc z)\,. \qquad (\text{df}\,\mathbf{Sum})$$

The definition of **Sum** is not a first-order formula.

Mereological sum

Fact

There is no sum of the empty set. This may be treated as strict formulation of the following statement: there is no empty mereological set.

Formally:

$$\neg \exists_{x \in M} x \ \mathbf{Sum} \ \emptyset \,. \tag{15}$$

Proof.

We use (df **Sum**):

- $\forall_{z \in \emptyset} z \sqsubseteq x$ – trivially true but
- $x \sqsubseteq x$, so $\exists_{z \in \emptyset} z \bigcirc x$, which is obviously false.

□

Mereological sum

Basic properties

Fact

$$\forall_{x \in M} x \ \mathbf{Sum}\ \{x\} .$$

Proof.

- $x \sqsubseteq x$
- $\forall_{z \in M}(z \sqsubseteq x \implies z \bigcirc x)$.

$\square$

Mereological sum

Basic properties

Fact

$$\forall_{x \in M} x \ \mathbf{Sum}\ \{y \in M \mid y \sqsubseteq x\}.$$

Proof.

- It is obvious that every element of $\{y \in M \mid y \sqsubseteq x\}$ is ingrediens of x.
- Take arbitrary z from the domain for which $z \sqsubseteq x$.
- So $z \in \{y \in M \mid y \sqsubseteq x\}$
- Since $z \bigcirc z$, so there is an element of $\{y \in M \mid y \sqsubseteq x\}$ which overlaps z (that is, z itself).

□

Mereological sum

Basic properties

Fact

If x has any parts (it is not a mereological atom), then it is a sum of its parts.

$$\forall_{x \in M}(\{y \in M \mid y \sqsubset x\} \neq \emptyset \Longrightarrow x \ \mathbf{Sum} \ \{y \in M \mid y \sqsubset x\}) .$$

Proof.

- Every element of $\{y \in M \mid y \sqsubset x\}$ is ingrediens of x.
- Take arbitrary z from the domain for which $z \sqsubseteq x$.
- We have two possibilities: either $z \sqsubset x$ or $z = x$
- Consider the first one. In such case $z \in \{y \in M \mid y \sqsubset x\}$ and $z \bigcirc x$.
- Consider the second one. In this case any element of $\{y \in M \mid y \sqsubset x\}$ will do.

□

Uniqueness of mereological sum

How many mereological sums of a given set exist?

It depends!

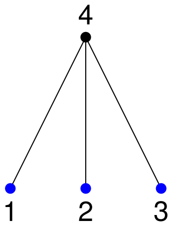
Depends on what?

It depends!

Mereological sum does not have to exist at all

It does not follow from (A), (T), (SSP) and (df **Sum**) that every subset of the domain has its mereological sum.

Model



- Consider $\{1, 2\}$.
- $3 \sqsubseteq 4$ but $1 \wr 3$ and $2 \wr 3$, so $\neg 3$ **Sum** $\{1, 2\}$.
- Nor any other object is a sum of $\{1, 2\}$.
- Similarly, $\{1, 3\}$ and $\{2, 3\}$ does not have its sums.

There may be more than one mereological sum

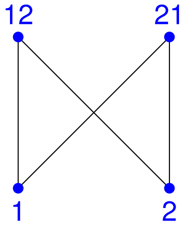
It does not follow from (A), (T) and (df **Sum**) that if a given set has a sum, then this sum is unique.

Model

- 0 **Sum** $\{0\}$,
- 1 **Sum** $\{0\}$,
- $0 \neq 1$.

There may be more than one mereological sum

Model



- 12 **Sum** $\{1, 2\}$
- 21 **Sum** $\{2, 1\}$
- $12 \neq 21$.

Uniqueness of mereological sum

Lemma

By (T) *and* (SSP), *for all* $a, b \in M$ *and all* $X \in \mathcal{P}(M)$ *it is the case that:*

$$\left(\forall_{z \in M}(z \sqsubseteq a \implies \exists_{x \in X} x \bigcirc z) \land X \subseteq Y \land \forall_{y \in Y} y \sqsubseteq b\right) \implies a \sqsubseteq b\,. \tag{16}$$

Proof.

- Assume (a) $\forall_{z \in M}(z \sqsubseteq a \implies \exists_{x \in X} x \bigcirc z)$, (b) $X \subseteq Y$ and (c) $\forall_{y \in Y} y \sqsubseteq b$.
- Take arbitrary $z \sqsubseteq a$. Thus by (a) we have $\exists_{x \in X} x \bigcirc z$.
- Let x_0 be that object: $x_0 \bigcirc z$.
- Since $x_0 \in X$, by (b) and (c) it is the case that $x_0 \sqsubseteq b$.
- Thus $z \bigcirc b$. That is $z \sqsubseteq a \implies z \bigcirc b$.
- But z was arbitrary, so $\forall_{z \in M}(z \sqsubseteq a \implies z \bigcirc b)$.
- So from (SSP) we obtain that $a \sqsubseteq b$.

Uniqueness of mereological sum

Theorem

It follows from axioms (A), (T), (SSP) *and* (df **Sum**) *that if a given set of objects has a sum, then it is unique:*

$$\forall_{a \in M} \forall_{b \in M} \forall_{X \in \mathcal{P}(M)} (a \textbf{ Sum } X \wedge b \textbf{ Sum } X \Longrightarrow a = b) .$$

Proof.

- Assume (1) a **Sum** X and (2) b **Sum** X.
- From (1) and (df **Sum**) we get: $\forall_{z \in M}(z \sqsubseteq a \Longrightarrow \exists_{x \in X} z \bigcirc x)$.
- From (2) and (df **Sum**) we get: $\forall_{x \in X} x \sqsubseteq b$.
- Applying the lemma from previous side we get: $a \sqsubseteq b$.
- We show similarly that $b \sqsubseteq a$.
- Using antisymmetry of $\sqsubseteq$ we have that $a = b$.

□

Classical mereology

- We are going to assume the strongest axiom concerning existence of mereological sum: every nonempty subset of the domain has its mereological sum.

$$\forall_{X \in \mathcal{P}_+(M)} \exists_{x \in M}\, x \ \textbf{Sum}\ X\,. \tag{EM}$$

- Such a solution seems to be the best from the point of view of applications mereology in point-free geometry and topology.
- By the classical mereology we mean any theory which is equivalent to the theory whose axioms are: (A), (T), (SSP) and (EM).
- Let **CM** be the class of all classical mereological structures, that is

$$\mathbf{CM} := \left\{\langle M, \sqsubset\rangle \mid \langle M, \sqsubset\rangle \text{ satisfies (A), (T), (SSP) and (EM)}\right\}. \tag{df $\mathbf{CM}$}$$

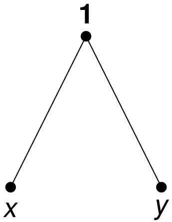
Existence of the unity

- Convention: for simplicity I will use the expression 'mereological structure' meaning classical mereological structure.
- Suppose that $\langle M, \sqsubset \rangle \in \mathbf{CM}$. By (EM) it is the case that there is (the unique) mereological sum of the set M. We will call it the unity of a structure.
- Formally, by means of the description operator, we define the unity as follows:

$$\mathbf{1} := (\iota x)\, x \ \mathbf{Sum}\ M\,. \qquad \text{(df } \mathbf{1}\text{)}$$

- Trivially, $\forall_{x \in M} x \sqsubseteq \mathbf{1}$.

A simple example of mereological structure



- It is the simplest non-degenerate structure.
- There is no structure with two elements.
- More complicated structures to follow.

Mereological sum vs. supremum

Fact

It follows from ($T_{\sqsubseteq}$) *and* (SSP) *that:*

$$\mathbf{Sum} \subseteq \mathbf{Sup}\,. \tag{17}$$

We will use the following (already proven) lemma:

$$\Big(\forall_{z\in M}(z \sqsubseteq a \implies \exists_{x\in X} x \bigcirc z) \wedge X \subseteq Y \wedge \forall_{y\in Y} y \sqsubseteq b\Big) \implies a \sqsubseteq b\,.$$

Proof.

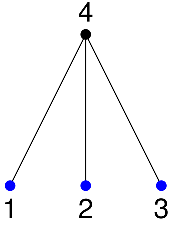
- Suppose that x **Sum** X, that is (a) $\forall_{y\in X} y \sqsubseteq x$ and (b) $\forall_{a\in M}(a \sqsubseteq x \implies \exists_{y\in X} y \bigcirc a)$.
- Take $b \in M$ and let $\forall_{y\in X} y \sqsubseteq b$.
- Now apply the lemma in question and conclude that $x \sqsubseteq b$.
- Thus x **Sup** X.

□

Mereological sum vs. supremum

The axioms (A), (T), (SSP) are to weak to prove that **Sup** $\subseteq$ **Sum**. Consider the model below.

Model



- Consider $\{1, 2\}$.
- 4 **Sup** $\{1, 2\}$.
- $\neg$4 **Sup** $\{1, 2\}$.
- Similarly, $\{1, 3\}$ and $\{2, 3\}$ do not have sums but have suprema.

Mereological sum vs. supremum

Let me remind that:

$$a \ \mathbf{Sup}\ X \wedge b \ \mathbf{Sup}\ X \Longrightarrow a = b\,. \qquad (\mathbf{Sup}!)$$

Fact

If a given structure satisfies ($\mathrm{T}_{\sqsubseteq}$) *and* (SSP), *then:*

$$a \ \mathbf{Sup}\ X \wedge b \ \mathbf{Sum}\ X \Longrightarrow a = b\,. \qquad (18)$$

Proof.

- We have proven that **Sum** $\subseteq$ **Sup**.
- Thus it is enough to apply the fact above.

$\square$

Mereological sum vs. supremum

Theorem

In every mereological structure:

$$x \ \mathbf{Sum}\ X \iff X \neq \emptyset \wedge x \ \mathbf{Sup}\ X\,.$$

Proof.

- The implication from left to right has already been proven.
- Assume $X \neq \emptyset$ and x **Sup** X.
- By (EM) there is y such that y **Sum** X.
- So it must be the case that $x = y$ and x **Sum** X, as required.

$\square$

Mereological sum vs. supremum

Conclusions

In every mereological structure:

$$\forall_{X \in \mathcal{P}_+(M)} (\iota x)\, x \ \mathbf{Sum}\ X = (\iota x)\, x \ \mathbf{Sup}\ X\,, \tag{19}$$

$$\mathrm{Card}(M) > 1 \Longrightarrow \neg \exists_{x \in M} x \ \mathbf{Sup}\ \emptyset\,, \tag{20}$$

$$\mathrm{Card}(M) > 1 \Longrightarrow \mathbf{Sum} = \mathbf{Sup}\,. \tag{21}$$

Algebraic operations

- The partial operation of merological sum $\bigsqcup\colon \mathcal{P}_+(M) \longrightarrow M$ such that:

$$\bigsqcup X := (\iota x)\, x \,\textbf{Sum}\, X\,. \tag{df $\bigsqcup$}$$

- The binary operation $\sqcup\colon M^2 \longrightarrow M$ such that:

$$x \sqcup y := \bigsqcup\{x, y\}\,. \tag{df $\sqcup$}$$

- Basic properties:

$$x \sqcup y = y \sqcup x\,, \tag{22}$$

$$(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)\,, \tag{23}$$

$$x = x \sqcup x\,, \tag{24}$$

$$x \sqsubseteq x \sqcup y\,. \tag{25}$$

Algebraic operations

- The partial operation of merological product $\sqcap\colon \{\langle x, y\rangle \mid x \bigcirc y\} \longrightarrow M$ such that:

$$x \sqcap y := \bigsqcup \{a \in M \mid a \sqsubseteq x \wedge a \sqsubseteq y\}\,. \tag{df $\sqcap$}$$

- Basic properties:

$$x \sqcap y := (\iota z)\, z \,\mathbf{Inf}\, \{x, y\}\,, \tag{26}$$

$$x \sqcap y = y \sqcap x\,, \tag{27}$$

$$(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)\,, \tag{28}$$

$$x = x \sqcap x\,, \tag{29}$$

$$x \sqsubseteq y \Longrightarrow x \sqcap y = x\,, \tag{30}$$

$$x \sqsubseteq y \sqcap z \iff x \sqsubseteq y \wedge x \sqsubseteq z\,. \tag{31}$$

Algebraic operations

- The partial operation of mereological complement $- : M \setminus \{\mathbf{1}\} \longrightarrow M$ such that:

$$- x := \bigsqcup \{ y \in M \mid y \wr x \} . \tag{df $-$}$$

- Basic properties:

$$x \neq \mathbf{1} \Longrightarrow (x \sqcup - x = \mathbf{1}) , \tag{32}$$

$$x \neq \mathbf{1} \Longrightarrow - - x = x , \tag{33}$$

$$x \neq \mathbf{1} \Longrightarrow (x \wr y \iff x \sqsubseteq - y) , \tag{34}$$

$$x \neq \mathbf{1} \wedge y \neq \mathbf{1} \Longrightarrow (x \sqsubseteq y \iff - y \sqsubseteq - x) . \tag{35}$$

Algebraic operations

- The partial binary operation of relative mereological complement such that:

$$y \sqsubset x \Longrightarrow x - y := \bigsqcup \{z \in M \mid z \sqsubseteq x \wedge z \wr y\}\,. \tag{df –}$$

- Basic properties:

$$y \sqsubset x \Longrightarrow x - y = x \sqcap -y\,, \tag{36}$$

$$y \sqsubset x \Longrightarrow (x - y) \wr y\,, \tag{37}$$

$$x \neq \mathbf{1} \Longrightarrow -x = \mathbf{1} - x\,. \tag{38}$$

Classical mereology and Boolean algebras

Theorem (Tarski)

Let $\mathfrak{B} = \langle A, +, \cdot, -, \mathbf{0}, \mathbf{1} \rangle$ *be a non-degenerate complete Boolean algebra and* $\langle A, \leqslant, \mathbf{0}, \mathbf{1} \rangle$ *be the Boolean lattice for* $\mathfrak{B}$*. Let:*

$$\sqsubseteq \; := \; \leqslant|_{A \setminus \{\mathbf{0}\}} \, .$$

Then $\langle A \setminus \{\mathbf{0}\}, \sqsubseteq \rangle \in \mathbf{CM}$.

Classical mereology and Boolean algebras

Theorem (Tarski)

Let $\langle M, \sqsubseteq \rangle \in \mathbf{CM}$ and let $\mathbf{0}$ be an arbitrary object such that $\mathbf{0} \notin M$. Let $\mathbf{1}$ be the unity of $\langle M, \sqsubseteq \rangle$. For $x, y \in M$ define:

$$x \oplus y := x \sqcup y$$

$$x \oplus \mathbf{0} := \mathbf{0} \oplus x := x \quad \textit{and} \quad \mathbf{0} \oplus \mathbf{0} := \mathbf{0}$$

$$x \bigcirc y \Longrightarrow x \odot y := x \sqcap y$$

$$x \wr y \Longrightarrow x \odot y := \mathbf{0}$$

$$x \odot \mathbf{0} := \mathbf{0} \odot x := \mathbf{0} \quad \textit{and} \quad \mathbf{0} \odot \mathbf{0} := \mathbf{0}$$

$$x \neq \mathbf{1} \Longrightarrow \ominus x := -x$$

$$\ominus \mathbf{1} := \mathbf{0} \quad \textit{and} \quad \ominus \mathbf{0} := \mathbf{1}.$$

Then, the structure $\langle M \cup \{\mathbf{0}\}, \oplus, \odot, \ominus, \mathbf{0}, \mathbf{1} \rangle$ is a complete Boolean algebra in which the standard order relation $\leqslant$ satisfies the following condition:

$$x \leqslant y \iff x \sqsubseteq y \vee x = \mathbf{0}.$$

The End
of
Part II