Point-free geometry and topology
Part I: Introduction

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ESSLLI 2012

Outline

Posets and lattices

Definition (of a poset)

A pair $\langle A, \leqslant \rangle$, where A is a set and $\leqslant \subseteq A \times A$, is a partially ordered set (abbr. poset) iff $\leqslant$ is reflexive, antisymmetrical and transitive:

$$\forall_{a \in A}\, a \leqslant a\,, \tag{1}$$

$$\forall_{a,b \in A} (a \leqslant b \wedge b \leqslant a \Longrightarrow a = b)\,, \tag{2}$$

$$\forall_{a,b,c \in A} (a \leqslant b \wedge b \leqslant c \Longrightarrow a \leqslant c)\,. \tag{3}$$

Definition (of upper and lower bound)

Let $\langle A, \leqslant \rangle$ be a poset and $X \subseteq A$. An element $a \in A$ is said to be an upper bound of X iff

$$\forall_{x \in X} x \leqslant a\,,$$

or a lower bound of X iff

$$\forall_{x \in X} a \leqslant x\,.$$

Definition (of supremum and infimum)

Let $\langle A, \leqslant \rangle$ be a poset and $X \subseteq A$. An element $a \in A$ is said to be the supremum of X iff is is the smallest upper bound of X, that is

$$\forall_{x \in X} x \leqslant a \wedge \forall_{b \in A} (\forall_{x \in X} x \leqslant b \implies a \leqslant b)\,.$$

An element $a \in A$ is said to be the infimum of X iff is is the greatest lower bound of X, that is

$$\forall_{x \in X} a \leqslant x \wedge \forall_{b \in A} (\forall_{x \in X} b \leqslant x \implies b \leqslant a)\,.$$

The supremum and infimum of X (if exist) will be denoted by '**Sup** X' and '**Inf** X'.

Posets and lattices

Definition (of a (complete) lattice)

A poset $\langle A, \leqslant \rangle$ is a lattice iff for any $a, b \in A$ there exist both the supremum and the infimum of $\{a, b\}$. In every lattice we can introduce two binary operations of meet and join:

$$a \vee b := \textbf{Sup}\{a, b\}\,, \tag{df $\vee$}$$

$$a \wedge b := \textbf{Inf}\,\{a, b\}\,. \tag{df $\wedge$}$$

A lattice is complete iff every its subset has the supremum and the infimum.

Topological spaces

Definition (of a topological space)

A pair $\langle X, \mathscr{O} \rangle$ where X is a nonempty set and $\mathscr{O} \subseteq \mathcal{P}(X)$ is a topological space iff:

$$\emptyset, X \in \mathscr{O}\,, \tag{4}$$

$$X_1, X_2 \in \mathscr{O} \Longrightarrow X_1 \cap X_2 \in \mathscr{O}\,, \tag{5}$$

$$\mathscr{X} \subseteq \mathscr{O} \Longrightarrow \bigcup \mathscr{X} \in \mathscr{O}\,. \tag{6}$$

We will often refer to $\mathscr{O}$ as topology on the set X.
We take $\mathbf{Int}\colon \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ to be the standard topological interior operation:

$$\mathbf{Int}(A) := \bigcup \{Y \in \mathscr{O} \mid Y \subseteq A\}\,. \tag{df $\mathbf{Int}$}$$

Topological spaces

Let $\langle X, \mathscr{O} \rangle$ be a topological space. The pair $\langle \mathscr{O}, \subseteq \rangle$ is a complete lattice with:

$$\begin{aligned}
O_1 \vee O_2 &= O_1 \cup O_2 \,, \\
O_1 \wedge O_2 &= O_1 \cap O_2 \,, \\
\mathscr{X} \subseteq \mathscr{O} \Longrightarrow \textbf{Sup}\, \mathscr{X} &= \bigcup \mathscr{X} \,, \\
\mathscr{X} \subseteq \mathscr{O} \Longrightarrow \textbf{Inf}\, \mathscr{X} &= \textbf{Int}(\bigcap \mathscr{X}) \,.
\end{aligned}$$

This lattice satisfies an infinite distributive law:

$$O \wedge \textbf{Sup}\, \mathscr{X} = \textbf{Sup}\{O \wedge P \mid P \in \mathscr{X}\} \,.$$

Frames

A frame is any complete lattice $\langle A, \leqslant \rangle$ satisfying the infinite distributive law:

$$a \wedge \mathbf{Sup}\, B = \mathbf{Sup}\{a \wedge b \mid b \in B\}\,. \tag{7}$$

Roughly speaking, in one sense point-free topology is studying topology via studying frames and their properties (that is without direct reference to points as primitive objects of the theory).

Important — our approach is different.

Three aspects of geometry

- Practical,
- mathematical,
- metamathematical.

The practical aspect of geometry

- From practical point of view geometry can be seen as a branch of applied science.
- It is used by physicists to build and develop theories in physics (both applied and theoretical), and by engineers to raise constructions like buildings and bridges.
- As its name indicates geometry is about measuring and this latest activity is crucial in both examining and transforming the world that surrounds us.

The mathematical aspect of geometry

- As a mathematical theory geometry can be viewed as a formal theory: a collection of basic concepts and axioms built upon first or second order logic that are simply a peculiar kind of machinery to produce strings of symbols that we usually call theorems.
- A little bit different approach is model theoretic one, in which we treat geometry as a theory of relational structures. Its aim is to reveal as many as possible properties of such structures.
- Thus for example the Pythagorean theorem is a result of using geometry as a mathematical theory.

The metamathematical aspect of geometry

- Metamathematical aspect concerns questions about geometry as a mathematical theory (in both approaches, formal and model theoretic).
- Such questions may be about primitive concepts, relations between them, independence of axioms, possibility of some constructions, models of axiomatic systems of geometry.
- Thus proving that the parallel axiom is independent from the remaining ones is a metamathematical theorem, and such is any proof of the impossibility of squaring the circle.
- Among metamathematical questions we can distinguish those that concern ontology of geometry as a mathematical theory. Thus in this particular case we simply ask: is it necessary to take points as first-order objects of geometry or can they be replaced with some other entities?

Set theoretical approach to point-based geometry

Before we go on to analyze point-free systems of geometry and topology, we will remind what is the usual set theoretical approach to geometry.

Set theoretical approach to point-based geometry

Definition (of a relational structure)

- A pure relational structure is any tuple $\langle \mathbf{D}, (R_i)_{i\in I}\rangle$, where $\mathbf{D}$ is a domain while for every $i \in I$, R_i is a relation in $\mathbf{D}$ or in a power set of $\mathbf{D}$ or is a hybrid relation, that is their elements are for example in $\mathbf{D} \times \mathcal{P}(\mathbf{D})$.
- If for any R_i there is a natural number n such that $n \geqslant 1$ and $R_i \subseteq \mathbf{D}^n$, then we say that R_i is an elementary relation. If every R_i is elementary, then we call $\langle \mathbf{D}, (R_i)_{i\in I}\rangle$ an elementary relational structure.
- Those structures that do not satisfy this condition are called non-elementary ones.

Set theoretical approach to point-based geometry

- In set theoretical approach to geometry we deal with some pure relational structures.
- A domain of a structure is a set of all points, which is called space. We will denote such a set by means of letter '**P**'. Other primitive notions of such a structure can be elementary or not.

Set theoretical approach to point-based geometry

In Foundations of Geometry by K. Borsuk and W. Szmielew, with reference to David Hilbert's book of the same title, the authors examine structures of the form $\langle \mathbf{P}, \mathfrak{L}, \mathfrak{P}, \mathbf{B}, \mathbf{D} \rangle$, in which:

- $\mathbf{P}$ is a non-empty set of points,
- $\mathfrak{L}$ and $\mathfrak{P}$ are subsets of $\mathscr{P}(\mathbf{P})$ (thus these notions are non-elementary, so structures examined are non-elementary as well),
- $\mathbf{B}$ and $\mathbf{D}$ are, respectively, ternary and quaternary relation in $\mathbf{P}$.
- Elements of $\mathfrak{L}$ and $\mathfrak{P}$ are called, respectively, lines and planes, $\mathbf{B}$ is called betweenness relation and $\mathbf{D}$ equidistance relation.
- We put specific axioms on $\mathbf{P}$, $\mathfrak{L}$, $\mathfrak{P}$, $\mathbf{B}$ and $\mathbf{D}$, and in this way we obtain a system of geometry that would probably satisfy Euclid and his contemporaries.

Set theoretical approach to point-based geometry

We can modify the above approach to start with structures $\langle \mathbf{P}, \mathbf{B}, \mathbf{D} \rangle$ and subsequently take such a collection of axioms that $\mathfrak{L}$ and $\mathfrak{P}$ will be definable be means of $\mathbf{B}$.
The set of lines can be defined in the following way

$$X \in \mathfrak{L} \stackrel{\text{df}}{\Longleftrightarrow} \exists_{p,q \in \mathbf{P}} (p \neq q \wedge$$
$$X = \{r \in \mathbf{P} \mid \langle r, p, q \rangle \in \mathbf{B} \vee \langle p, r, q \rangle \in \mathbf{B} \vee \langle p, q, r \rangle \in \mathbf{B}\} \cup \{p, q\}),$$

where the condition '$\langle r, p, q \rangle \in \mathbf{B}$' says that point p is between points q and r.

Set theoretical approach to point-based geometry

To define $\mathfrak{P}$, first we introduce a new relation $\mathbf{L} \subseteq \mathbf{P}^3$, so called relation of collinearity of points

$$\langle p, q, r\rangle \in \mathbf{L} \overset{\text{df}}{\Longleftrightarrow} \exists_{X \in \mathfrak{L}}(p \in X \wedge q \in X \wedge r \in X).$$

Subsequently we define a triangle, whose cones are located in three points p, q, r (in symbols '$\mathrm{tr}(pqr)$') that are not collinear

$$\begin{aligned}\neg\mathbf{L}(p, q, r) \Longrightarrow &\\ &\mathrm{tr}(pqr) := \{a \in \mathbf{P} \mid a = p \vee a = q \vee a = r \vee \\ &\qquad \langle p, a, q\rangle \in \mathbf{B} \vee \langle p, a, r\rangle \in \mathbf{B} \vee \langle q, a, r\rangle \in \mathbf{B}\}.\end{aligned}$$

Now we define a plane

$$\begin{aligned}X \in \mathfrak{P} \overset{\text{df}}{\Longleftrightarrow} \exists_{p,q,r \in \mathbf{P}}\Big[\neg\mathbf{L}(p, q, r) \wedge X = \Big\{c \in \mathbf{P} \mid &\\ \exists_{a,b \in \mathbf{P}}[a \neq b \wedge a, b \in \mathrm{tr}(pqr) \wedge \langle c, a, b\rangle \in \mathbf{B} \vee \langle a, c, b\rangle \in \mathbf{B}]\Big\}\Big].\end{aligned}$$

Set theoretical approach to point-based geometry

Thus, in light of the above constructions, we conclude that to construct Euclidean geometry one can do with just three primitive notions: of point, of betweenness relation and of equidistance relation.

Set theoretical approach to point-based geometry

Mario Pieri (1860-1913)

- La geometria elementare istituita sulle nozioni “punto” é “sfera”, *Matematica e di Fisica della Società Italiana delle Scienze*, **vol. 15**, 1908, 345–450.
- In Polish: *Geometrja elementarna oparta na pojęciach „punktu” i „sfery”*, Gebether i Wolff, Warsaw, 1915.

Set theoretical approach to point-based geometry

- It was proven by Pieri that to construct a system of Euclidean geometry one actually needs only two primitive notions: that of point and that of equidistance relation, which in the Pieri's system case is a ternary relation among points.
- Denoting this relation by means of '$\triangle$' we can say that while doing geometry in Pieri's manner we analyze elementary structures $\langle \mathbf{P}, \triangle \rangle$, where $\triangle \subseteq \mathbf{P}^3$.
- Now we of course have to choose axioms to define $\mathfrak{L}$, $\mathfrak{P}$, **B** and **D** in such a way to be able to prove that this approach is definitionally equivalent to Hilbert's one (see R. Gruszczyński, A. Pietruszczak *Pieri's structures* in: Fundamenta Informaticae, 2007).

Set theoretical approach to point-based geometry

Theorem

All Pieri's structures are isomorphic, for any Pieri's structure $\langle \mathbf{P}, \Delta \rangle$ is isomorphic to $\langle \mathbb{R}^3, \Delta^{\mathbb{R}^3} \rangle$, where $\Delta^{\mathbb{R}^3}$ is introduced by means of the following definition

$$\bar{x}\bar{y} \ \Delta^{\mathbb{R}^3} \ \bar{z} \overset{\text{df}}{\iff} \varrho(\bar{x}, \bar{z}) = \varrho(\bar{y}, \bar{z}) \,, \qquad (\text{def } \Delta^{\mathbb{R}^3})$$

where $\bar{x}, \bar{y}, \bar{z} \in \mathbb{R}^3$ and $\varrho\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is the standard Euclidean metric.

Euclid’s Elements

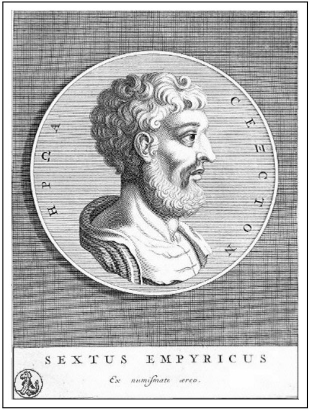
A point is that which has position but not dimensions.

The above definition is a starting point for the construction of Euclid’s system of geometry.

Against the Geometers

The main objection against points: they are treated as an ultimate constituent of reality, while we do not experience any objects that bear any resemblance to them.

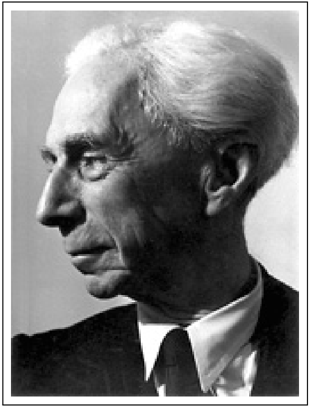
Sextus Empiricus (c. 160-210 AD)
Against the Geometers in: Against the Professors



Origins of point-free geometry

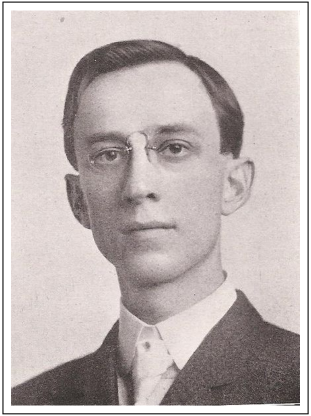
Bertrand Russell (1872–1970)

- Our Knowledge of The External World (1914)
- the perspective space
- no one has ever seen or touched a point



Origins of point-free geometry

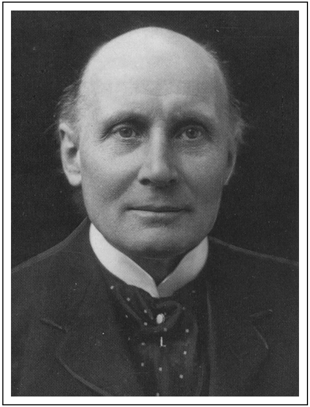
Theodore de Laguna Point, Line, and Surface, as Sets of Solids
Journal of Philosophy (1922)



Origins of point-free geometry

Alfred Whitehead (1861–1947)

- The Concept of Nature (1920)
- Process and Reality (1929)



Objections against classical point-based geometry

- Points, from which in a geometrical or a physical model space is built, are neither sensually experienced nor its existence can be derived from data (both by some experiment or some kind of reasoning); moreover we cannot point to objects in the real world, that could be «natural» counterparts of points.
- The space of geometry and its «parts» as distributive sets are abstract and as such they cannot be experienced empirically; the perspective space and its parts are concrete (sensually experienced).
- All objects that exist in the perspective space have dimensions and parts, so points cannot be elements of this space.

Objections against classical point-based geometry

- The problems described above were a stimulus to search for some other, different from point-based one, approach to geometry. Those approaches are usually named point-free or pointless.
- Those geometries do not either aim at replacing classical geometry with some other formal science or question usefulness of the notion of point. The introduction of this notion to science by the ancients was ingenious and enabled really impressive development of both mathematics and physics.

Point-free but not without points!

- The names ‘point-free’ and ‘pointless’ are a bit misguiding here!
- The crucial difference: point-free geometry does not have the notion of point among its primitive notions but it is defined by means of other primitive notions which intuitive interpretation is less problematic.

Point-free but not without points!

- In light of what has been said so far—point-free geometry looks for such foundations for classical, point-based geometry which are most satisfying from a point of view of our intuitions and representations concerning the perspective space.
- Point-free geometry still talks about points but the difference is that these are abstract objects constructed from objects that can be found in the perspective space. Points as such objects are still to behave like those in classical geometry and standard geometrical relations are to hold among them.
- Points as constructed from spatial objects do not have dimensions in this sense like the perspective space and its parts have, since they are not spatial at all. Therefore they satisfy, in a way, the Euclid’s definition.

The End
of
Part I