

Point-free geometry and topology

Part III: Basic assumptions of point free geometry

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Outline

Philosophical and methodological preliminaries

Leśniewski's approach to geometry

Clarifying and tidying up the assumptions

The perspective space

- ▶ Both Euclidean geometry and physics (or at least some of its fragments) aspire to describe the most general properties of something that, after Bertrand Russell, can be called **the perspective space**.
- ▶ Russell writes about so called **private spaces**, assuming that to each one of our senses corresponds one space. Thus we have **the space of sight**, **the space of touch** and so on. Every human perspective is contained in each own private space depending on which out of our senses is currently used.

The perspective space

- ▶ There is **only one all-embracing space which includes all those perspectives**. However:
 - ▶ its existence cannot be either proven or tested empirically,
 - ▶ it can only be deduced from our sensations and experiences,
 - ▶ thus the perspective space is not something that is given, but an intellectual construction.
- ▶ To quote Russell

All that experience makes certain is the several spaces of the several senses correlated by empirically discovered laws. The one space may turn out to be valid as a logical construction, compounded of the several spaces, but there is no good reason to assume its independent metaphysical reality.
- ▶ For the sake of the goals of this lectures it is enough to understand the perspective space (or simply **the space**) as the sensually accessible world that surrounds us.

Russell on points

Russell's words from *Our Knowledge of the External World*

It is customary to think of points as simple and infinitely small, but geometry in no way demands that we should think of them in this way. All that is necessary for geometry is that they should have mutual relations possessing certain enumerated abstract properties, and it may be that an assemblage of data of sensation will serve this purpose.

The task of point-free geometry

It thus can be said that the task of point-free geometry is to construct such mathematical objects among which there hold the same relations as among «ordinary» points and which fulfill the following requirements

- ▶ their ontological status will be less problematic than in case of Euclidean points;
- ▶ its «building material», out of which they will be constructed, could be naturally and intuitively interpreted in the perspective space.

Leśniewski's approach to geometry

Let us assume that:

- ▶ \mathbf{s} is the space,
- ▶ Pt is a distributive set of points,
- ▶ F is a distributive set of figures,
- ▶ L is a distributive set of lines,
- ▶ P a distributive set of planes.

Remark

Here we use different symbols for these sets than earlier, since these are *different* sets indeed.

- ▶ The set of all points is no longer space.
- ▶ In classical geometry figures, lines and planes are distributive sets of points. On the other hand, in the case considered these are fusions of points.

Leśniewski's approach to geometry

Then we have that:

- (i) $\mathbf{s} \neq Pt$ (space is not the set of all points);
- (ii) $\mathbf{s} = \sqcup Pt$ (space is the fusion of all points);
- (iii) $\mathbf{s} \in F$ (space is one of figures);
- (iv) $x \in F$ and $x \neq \mathbf{s}$ iff $x \sqsubset \mathbf{s}$ (every figure which is different from space is its part and conversely, every part of space is a figure);
- (v) $Pt, L, P \subseteq F$ (all points, lines and planes are figures, therefore they are parts of space).

Leśniewski's approach to geometry

- ▶ **ADVANTAGE:** Neither space nor figures are any longer identified with distributive sets of points
- ▶ **DISADVANTAGE:** The space is «infested with» less than three-dimensional objects whose counterparts are not present in the perspective space.

Ontological commitments of point-free geometry

- ▶ Instead of the set of figures we have the set of objects that are called **solids**, **regions** or **spatial bodies**. Let **R** be the set of all regions.
- ▶ **R** is ordered by the ingreditens relation.
- ▶ The space **s** (if it is assumed to exist) is usually the unity of **R**
- ▶ Lines and planes are not elements of **R**. Intuitively, **R** contains only three-dimensional and «regular» parts of space.

Points as distributive sets of regions

- Points are either distributive sets of regions or distributive sets of sets of regions. Let Π be the set of all points. Then:

$$\Pi \subseteq \mathcal{P}(\mathbf{R}) \quad \text{or} \quad \Pi \subseteq \mathcal{P}(\mathcal{P}(\mathbf{R})).$$

- $\Pi \neq \mathbf{s}$ (the set of all points is not the space).

Figures as sets of points

- ▶ A figure is defined in a standard way, as a nonempty set of points:

$$\mathfrak{F} := \mathcal{P}_+(\Pi).$$

- ▶ The set of all points is a figure: $\Pi \in \mathfrak{F}$.
- ▶ But:

$$\Pi \cap \mathbf{R} = \emptyset = \Pi \cap \mathfrak{F},$$

that is points are neither regions nor abstract figures.

- ▶ Lines and planes, similarly as in classical geometry, are distributive sets of points: $\mathfrak{L} \cup \mathfrak{P} \subseteq \mathfrak{F}$.

A very little bit of type theory

- ▶ In point-based geometries \mathfrak{F} has the type $(*)$ in a hierarchy of types over the base set.
- ▶ In point-free approach it has either the type $((*))$ or $(((*)))$.

Summary

- (i) $\mathbf{s} \neq \Pi$;
- (ii) $\mathbf{s} \in \mathbf{R}$ and $\mathbf{s} \notin \mathfrak{F}$ (the space is one of regions and is not an «abstract» figure, that is it is not a distributive set of points);
- (iii) $x \in \mathbf{R}$ and $x \neq \mathbf{s}$ iff $x \sqsubset \mathbf{s}$ (every region which is different from the space is its part and conversely, every part of the space is a region);
- (iv) $\Pi \subseteq 2^{\mathbf{R}}$ or $\Pi \subseteq 2^{2^{\mathbf{R}}}$ and $\mathfrak{L}, \mathfrak{P} \subseteq \mathfrak{F}$ (all points are sets whose elements are regions or sets of regions; all lines and planes are abstract figures, but they are not parts of \mathbf{s}).

In light of the above remarks we can say that the conditions (iii)–(iv) are **natural assumptions** of point-free geometry.

The End of Part III