

Point-free geometry and topology

Part V: Tarski's geometry of solids

Rafał Gruszczyński

Department of Logic
Nicolaus Copernicus University
Toruń, Poland

Salerno 2011

Outline

Introduction

Definition of concentricity relation

Points and equidistance relation

Specific axioms of geometry of solids

Categoricity of geometry of solids

“Some years ago Leśniewski suggested the problem of establishing the foundations of the geometry of solids understanding by this term a system of geometry destitute of such geometrical figures as points, lines, and surfaces, and admitting as figures only solids—the intuitive correlates of open (or closed) regular sets of the three-dimensional Euclidean geometry. The specific character of such geometry of solids—in contrast to all point geometries—is shown in particular in the law according to which each figure contains another figure as a proper part.”

A. Tarski *Foundations of the geometry of solids*

First steps

After Tarski we will analyze structures of the form $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ such that

- (P0)
- (i) $\langle \mathbb{R}, \sqsubseteq \rangle$ is a mereological structure,
 - (ii) $\mathbb{B} \subseteq \mathbb{R}$.
- ▶ Elements of \mathbb{R} will be called **regions**.
 - ▶ Elements of \mathbb{B} will be called **mereological balls** (or simply **balls** in case it follows from the context that we refer to elements of \mathbb{B}).
 - ▶ The notions of **region**, **ball** and **being an ingrediens** are primitive notions of the analyzed theory.

Super-space and space

From the axiom of mereological sum existence it follows that

- ▶ there exists the unity of the structure $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$, that is a mereological sum (a supremum) of a set \mathbb{R} (symb. '**1**', **super-space**);
- ▶ there exists a mereological sum of a set \mathbb{B} (**the space**):

$$s := (\iota x) x \text{ **Sum** } \mathbb{B}. \quad (\text{df } s)$$

- ▶ $s \sqsubseteq \mathbf{1}$.

Solids

Definition

By a **solid** in a structure $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ we will understand the mereological sum of an arbitrary non-empty subset of a set \mathbb{B} :

$$s \in \mathcal{S} \stackrel{\text{df}}{\iff} \exists Z \subseteq \mathbb{B} (Z \neq \emptyset \wedge s \text{ **Sum** } Z). \quad (\text{df } \mathcal{S})$$

Fact

Since $s \text{ **Sum** } \mathbb{B}$, then s is «the largest» solid, that is:

$$\forall s \in \mathcal{S} \mathbb{S} \sqsubseteq s.$$

External tangency of balls

Definition

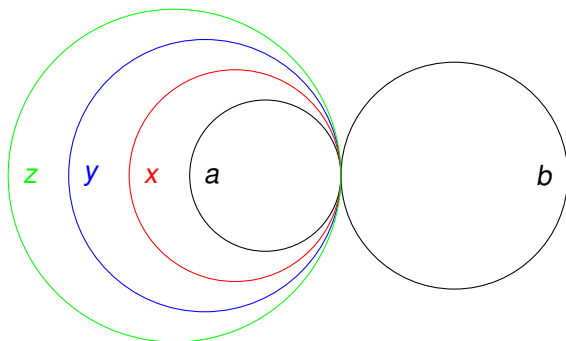
A ball a is **externally tangent** to a ball b iff

- (i) a is external to b and
- (ii) for any balls x and y such that a is ingreidiens of both x and y , while x and y are external to b , it is the case that x is an ingreidiens of y or y is an ingreidiens of x .

External tangency of balls

$$a \text{ ET } b \stackrel{\text{df}}{\iff} a, b \in \mathbb{B} \wedge$$

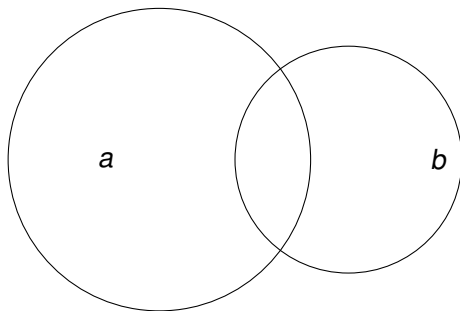
$$a \not\supseteq b \wedge \forall_{x,y \in \mathbb{B}} (a \sqsubseteq x \not\supseteq b \wedge a \sqsubseteq y \not\supseteq b \implies x \sqsubseteq y \vee y \sqsubseteq x).$$



External tangency of balls

$$a \text{ ET } b \stackrel{\text{df}}{\iff} a, b \in \mathbb{B} \wedge$$

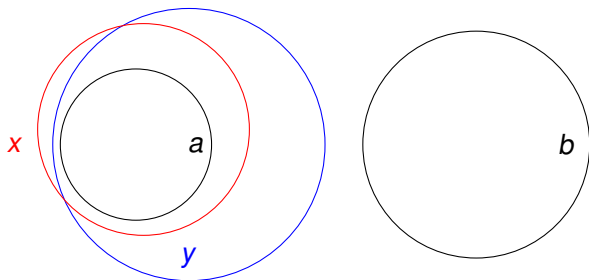
$$a \not\subset b \wedge \forall_{x, y \in \mathbb{B}} (a \sqsubseteq x \not\subset b \wedge a \sqsubseteq y \not\subset b \implies x \sqsubseteq y \vee y \sqsubseteq x).$$



External tangency of balls

$$a \text{ ET } b \stackrel{\text{df}}{\iff} a, b \in \mathbb{B} \wedge$$

$$a \not\subseteq b \wedge \forall_{x,y \in \mathbb{B}} (a \subseteq x \not\subseteq b \wedge a \subseteq y \not\subseteq b \implies x \subseteq y \vee y \subseteq x).$$



External tangency of balls – symmetry

- ▶ In case balls are balls of ordinary geometry then the relation of external tangency is of course symmetrical.
- ▶ It is worth noticing that it does not follow from the above definition and the axioms of mereology that $\mathbb{E}\mathbb{T}$ has the property of being symmetrical.

External tangency of balls – symmetry

Consider the mereological structure generated by four pairwise disjoint non-empty sets A , B , C and D :

$$\mathbb{R} := \{S \cup X \cup Y \cup Z \mid S, X, Y, Z \in \{A, B, C, D\}\}$$

and

$$\sqsubseteq := \subseteq \quad \text{and} \quad \mathbb{B} := \{A, B, A \cup C, A \cup D\}.$$

- ▶ Of course in this structure: $X \wr Y$ iff $X \cap Y = \emptyset$.
- ▶ Clearly, $B \mathbb{E} \top A$, since $B \wr A$ and there are no elements of \mathbb{B} different from B and such that B is their subset.
- ▶ But it is not the case that $A \mathbb{E} \top B$, since $A \sqsubseteq A \cup C$, $A \sqsubseteq A \cup D$, $B \wr A \cup C$, $B \wr A \cup D$, $A \cup C \not\sqsubseteq A \cup D$ and $A \cup D \not\sqsubseteq A \cup C$.

Internal tangency of balls

Definition

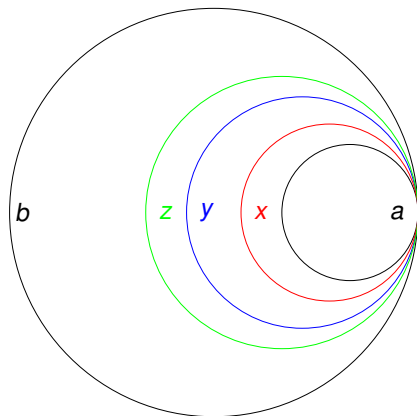
A ball a is **internally tangent** to a ball b iff

- (i) a is a part of b and
- (ii) for any balls x and y , of which a is an ingrediens and which are ingredienses of b , either x is an ingrediens of y or conversely.

Internal tangency of balls

$$a \text{ IT } b \stackrel{\text{df}}{\iff} a, b \in \mathbb{B} \wedge$$

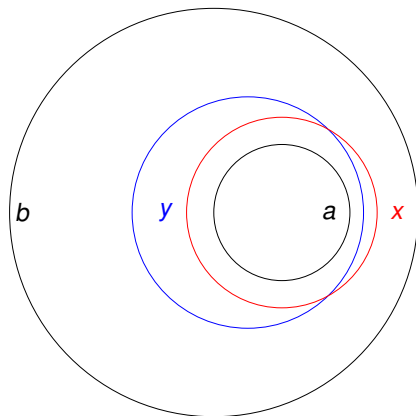
$$a \sqsubset b \wedge \forall_{x,y \in \mathbb{B}} (a \sqsubseteq x \sqsubseteq b \wedge a \sqsubseteq y \sqsubseteq b \implies x \sqsubseteq y \vee y \sqsubseteq x).$$



Internal tangency of balls

$$a \text{ IT } b \stackrel{\text{df}}{\iff} a, b \in \mathbb{B} \wedge$$

$$a \sqsubset b \wedge \forall_{x,y \in \mathbb{B}} (a \sqsubseteq x \sqsubseteq b \wedge a \sqsubseteq y \sqsubseteq b \implies x \sqsubseteq y \vee y \sqsubseteq x).$$



External diametrical tangency of balls

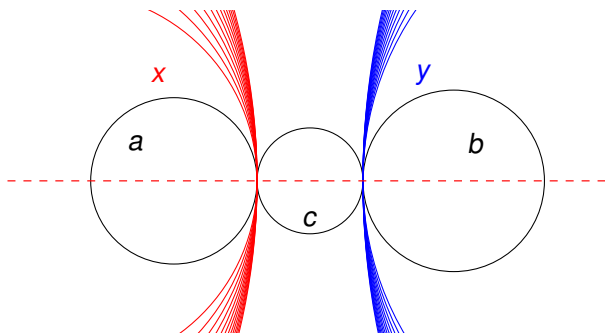
Definition

Balls a and b are **externally diametrically tangent** to a ball c iff

- (i) both a and b are externally tangent to c and
- (ii) for any balls x and y external to c and such that a is an ingrediens of x and b is an ingrediens of y , x is external to y .

External diametrical tangency of balls

$$ab \text{ EDC } c \stackrel{\text{df}}{\iff} a, b, c \in \mathbb{B} \wedge a \text{ ET } c \wedge b \text{ ET } c \wedge \\ \forall_{x, y \in \mathbb{B}} (a \sqsubseteq x \wr c \wedge b \sqsubseteq y \wr c \implies x \wr y).$$



Internal diametrical tangency of balls

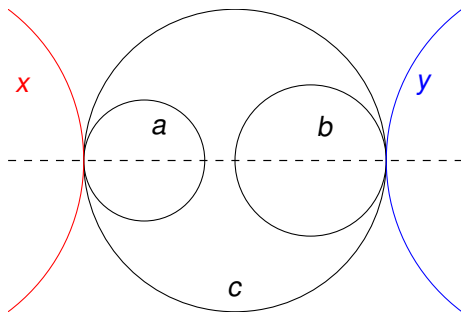
Definition

Balls a and b are **internally diametrically tangent** to a ball c iff

- (i) both a and b are internally tangent to c and
- (ii) for any balls x and y external to c and such that a is externally tangent to x and b is externally tangent to y , x is external to y .

Internal diametrical tangency of balls

$$ab \text{ ID } c \stackrel{\text{df}}{\iff} a, b, c \in \mathbb{B} \wedge a \text{ IT } c \wedge b \text{ IT } c \\ \wedge \forall x, y \in \mathbb{B} (x \text{ ? } c \wedge y \text{ ? } c \wedge a \text{ ET } x \wedge b \text{ ET } y \implies x \text{ ? } y).$$



Concentricity relation

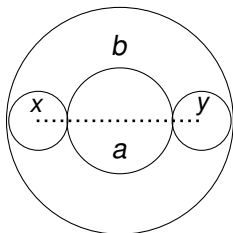
Definition

A ball a is **concentric** with a ball b iff there holds one of the following (mutually exclusive) conditions

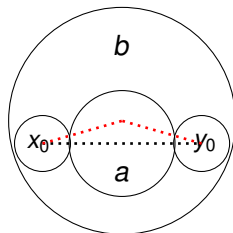
- (i) a is identical with b ;
- (ii) a is a part of b and for any balls x and y , which are both externally tangent to a and internally tangent to b , x and y are internally diametrically tangent to b ;
- (iii) b is a part of a and for any balls x and y , which are externally diametrically tangent to b and internally tangent to a , x and y are internally diametrically tangent to a .

Concentricity relation

$$a \odot b \stackrel{\text{df}}{\iff} a, b \in \mathbb{B} \wedge \left[a = b \right. \\ \left. \underline{\vee} (a \sqsubset b \wedge \forall_{x,y \in \mathbb{B}} (xy \in D a \wedge x \Vdash b \wedge y \Vdash b \implies xy \Vdash b)) \right. \\ \left. \underline{\vee} (b \sqsubset a \wedge \forall_{x,y \in \mathbb{B}} (xy \in D b \wedge x \Vdash a \wedge y \Vdash a \implies xy \Vdash a)) \right].$$



A



B

Definition of a point

Definition

By a **point** we mean the set of all those balls that are concentric with a given ball.

$$\beta \in \Pi \stackrel{\text{df}}{\iff} \exists_{b \in \mathbb{B}} \beta = \{x \in \mathbb{B} \mid x \odot b\}. \quad (\text{df } \Pi)$$

For any ball b , let π_b be the point determined by b , i.e.

$$\pi_b := \{x \in \mathbb{B} \mid x \odot b\}. \quad (\text{df } \pi_b)$$

By reflexivity of \odot we have:

$$\forall_{b \in \mathbb{B}} b \in \pi_b \in \Pi, \quad (1)$$

$$\forall_{\beta \in \Pi} \exists_{b \in \mathbb{B}} \beta = \pi_b, \quad (2)$$

$$\Pi \neq \emptyset \iff \mathbb{B} \neq \emptyset. \quad (3)$$

Equidistance relation among points

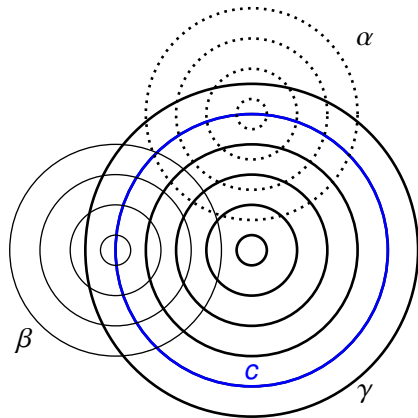
Definition

Points α and β are **equidistant** from a point γ iff

- (i) $\alpha = \beta = \gamma$ or
- (ii) there exists a ball in γ such that no ball from α or β either is an ingrediens of or is exterior to this ball.

Equidistance relation among points

$$\alpha\beta\Delta\gamma \stackrel{\text{df}}{\iff} \alpha = \beta = \gamma \vee \exists c \in \gamma \neg \exists a \in \alpha \cup \beta (a \sqsubseteq c \vee a \supseteq c). \quad (\text{df } \Delta)$$



First axiom

The first one of the specific axioms of the geometry of solids states that:

- ▶ points defined as sets of concentric balls are points of an ordinary point-based geometry,
- ▶ the relation Δ is an ordinary equidistance relation.

$\langle \Pi, \Delta \rangle$ is a Pieri's structure. (P1)

First axiom – consequences

Lemma

$\langle \Pi, \Delta \rangle$ is isomorphic to $\langle \mathbb{R}^3, \Delta^{\mathbb{R}^3} \rangle$, where for any $\bar{x}, \bar{y}, \bar{z} \in \mathbb{R}^3$:

$$\bar{x}\bar{y} \Delta^{\mathbb{R}^3} \bar{z} \stackrel{\text{df}}{\iff} \varrho(\bar{x}, \bar{z}) = \varrho(\bar{y}, \bar{z}).$$

By (P1) we have that the set of all mereological points has the power of continuum, i.e. $|\Pi| = \mathfrak{c}$. Hence, it is the case that

$$\mathfrak{c} = |\Pi| \leq |\mathbb{B}| \leq |\mathbb{S}| \leq |\mathbb{R}|.$$

Topology in the set of points

- ▶ Since, by (P1), $\langle \Pi, \Delta \rangle$ is a Pieri's structure, we can define the family of **all open balls** \mathbf{BO}_{Π} in $\langle \Pi, \Delta \rangle$.
- ▶ By means of it we can introduce in Π the family \mathcal{O}_{Π} of **open sets** and three standard topological operations \mathbf{Int}_{Π} , \mathbf{Cl}_{Π} and $\mathbf{Fr}_{\Pi}: 2^{\Pi} \rightarrow 2^{\Pi}$, where $\mathbf{Fr}_{\Pi} A := \mathbf{Cl}_{\Pi} A \setminus \mathbf{Int}_{\Pi} A$, for any $A \subseteq \Pi$.
- ▶ Thus $\langle \Pi, \mathcal{O}_{\Pi} \rangle$ is a **topological space**.
- ▶ Moreover, it has **all topological properties of 3-dimensional Euclidean space**.

Topology in the set of points

- ▶ $\langle \Pi, \mathcal{O}_\Pi \rangle$ has all topological properties of three-dimensional Euclidean space.
- ▶ In $\langle \Pi, \mathcal{O}_\Pi \rangle$ we introduce the family \mathbf{RO}_Π of all regular open sets (open domains), i.e. the family of sets that are equal to the interior of their closure.
- ▶ Let $\mathbf{RO}_\Pi^+ := \mathbf{RO}_\Pi \setminus \{\emptyset\}$.

Lemma

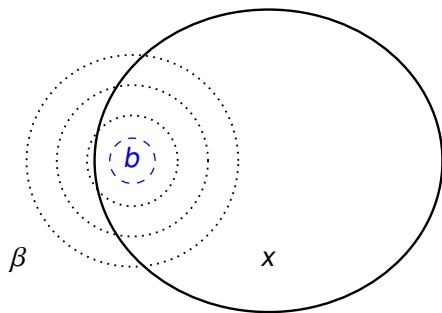
If $\langle \mathbb{R}, \mathbb{B}, \subseteq \rangle$ satisfies (P0) and (P1), then $\langle \mathbf{RO}_\Pi^+, \mathbf{BO}_\Pi, \subseteq \rangle$ is isomorphic to $\langle \mathbf{RO}_{\mathbb{R}^3}^+, \mathbf{BO}_{\mathbb{R}^3}, \subseteq \rangle$.

Interior points of regions

Definition

A point β is an **interior point** of a region x iff there exists a ball $b \in \beta$ such that $b \sqsubseteq x$:

$$\text{int}(x) := \{\beta \in \Pi \mid \exists b \in \beta \ b \sqsubseteq x\}. \quad (\text{df int})$$

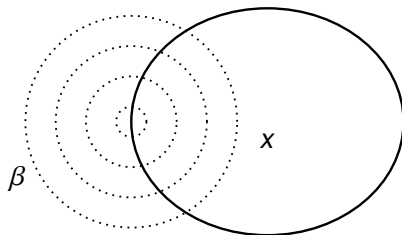


Fringe points of regions

Definition

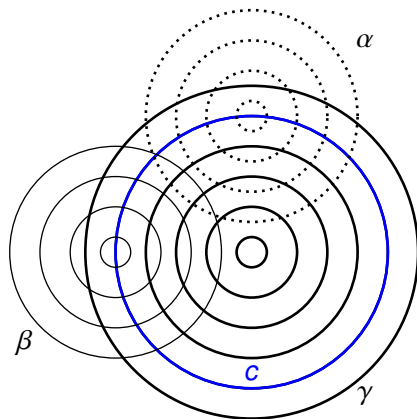
A point β is a **fringe point** of a region x from \mathbb{R} iff no ball in β is either ingredients of or is exterior to x :

$$\begin{aligned}\text{fr}(x) &:= \{\beta \in \Pi \mid \neg \exists b \in \beta (b \sqsubseteq x \vee b \upharpoonright x)\} \\ &= \{\beta \in \Pi \mid \forall b \in \beta (b \not\sqsubseteq x \wedge b \circ x)\}.\end{aligned}\quad (\text{df fr})$$



Equidistance – an intuitive explanation

$$\alpha\beta\Delta\gamma \iff \alpha = \beta = \gamma \vee \exists_{c \in \gamma} (\alpha \in \text{fr}(c) \wedge \beta \in \text{fr}(c)). \quad (\text{df' } \Delta)$$



The second axiom

The set of interior points of any solid is a non-empty open domain in Π :

$$\forall s \in \mathcal{S} \quad \text{int}(s) \in \mathbf{RO}_{\Pi}^+. \quad (\text{P2})$$

The third axiom

For any non-empty regular open subset of Π there exists a solid such that this subset is the set of all interior points of this solid:

$$\forall U \in \mathbf{RO}_{\Pi}^+ \exists s \in \mathcal{S} \text{ int}(s) = U. \quad (\text{P3})$$

The fourth axiom

For any solids s_1 and s_2 , if the set of all interior points of s_1 is a subset of the set of all interior points of s_2 , then s_1 is an ingrediens of s_2 , i.e.

$$\forall_{s_1, s_2 \in \mathcal{S}} (\text{int}(s_1) \subseteq \text{int}(s_2) \implies s_1 \sqsubseteq s_2). \quad (\text{P4})$$

The fifth (and last) axiom

Every region has a ball:

$$\forall x \in \mathbb{R} \exists b \in \mathbb{B} \ b \sqsubseteq x. \quad (\text{P5})$$

Axioms of geometry of solids – summary

- ▶ $\langle \mathbb{R}, \sqsubseteq \rangle$ is a mereological structure.
- ▶ $\langle \Pi, \Delta \rangle$ is a Pieri's structure.
- ▶ $\forall s \in \mathcal{S} \text{ int}(s) \in \mathbf{RO}_{\Pi}^+$.
- ▶ $\forall U \in \mathbf{RO}_{\Pi}^+ \exists s \in \mathcal{S} \text{ int}(s) = U$.
- ▶ $\forall s_1, s_2 \in \mathcal{S} (\text{int}(s_1) \subseteq \text{int}(s_2) \implies s_1 \sqsubseteq s_2)$.
- ▶ $\forall x \in \mathbb{R} \exists b \in \mathcal{B} b \sqsubseteq x$.

Categoricity

Theorem

Any structure $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ satisfying axioms (P0)–(P5) is isomorphic to $\langle \mathbf{RO}_{\mathbb{R}^3}^+, \mathbf{BO}_{\mathbb{R}^3}, \sqsubseteq \rangle$.

Lemma

The mapping $\text{int}: \mathbb{R} \rightarrow \mathbf{RO}_{\Pi}^+$ is an isomorphism from $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ onto $\langle \mathbf{RO}_{\Pi}^+, \mathbf{BO}_{\Pi}, \sqsubseteq \rangle$.

Theorem

For every $b \in \mathbb{B}$, $\text{int}(b) \in \mathbf{BO}_{\Pi}$.

Theorem

For every $B \in \mathbf{BO}_{\Pi}$ there is exactly one ball $b \in \mathbb{B}$ such that $\text{int}(b) = B$.

Categoricity

- ▶ The structures $\langle \mathbf{RO}_{\mathbb{R}^3}^+, \mathbf{BO}_{\mathbb{R}^3}, \subseteq \rangle$ and $\langle \mathbf{RC}_{\mathbb{R}^3}^+, \mathbf{BC}_{\mathbb{R}^3}, \subseteq \rangle$ are isomorphic.
- ▶ Thus $\langle \mathbb{R}, \mathbb{B}, \subseteq \rangle$ is isomorphic to both of them.
- ▶ So, from ontological point of view, «real» solids are neither open nor closed (in the sense of topology).
- ▶ The notions of **interior of solid** and of **border (fringing) of solid** are abstract—they do not refer to solids (first-order objects) but to abstract sets consisted of abstract points.

The End
of
Part V
(and the last one)

Thank you
for
your attention :)