Point-free geometry and topology
Part IV: Grzegorczyk's system

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Outline

Grzegorczyk's structures

Definition of a point in G-structures

Properties of points in G-structures

Topology in the set of points

Quasi-separation structures

Definition

By a quasi-separation structure we mean any structure $\langle \mathbb{R}, \sqsubseteq, \mathrel{)(} \rangle$ such that $\mathrel{)(} \subseteq \mathbb{R} \times \mathbb{R}$ (it is called a separation relation) and

(S0) $\langle \mathbb{R}, \sqsubseteq \rangle$ is a mereological structure,

(S1) $\forall_{x,y \in \mathbb{R}} \big(x \mathrel{)(} y \Longrightarrow x \not\sqsubseteq y \big)$,

(S2) $\forall_{x,y \in \mathbb{R}} \big(x \mathrel{)(} y \Longrightarrow y \mathrel{)(} x \big)$,

(S3) $\forall_{x,y,z \in \mathbb{R}} \big(x \sqsubseteq y \wedge z \mathrel{)(} y \Longrightarrow z \mathrel{)(} x \big)$.

Non-tangential inclusion of regions

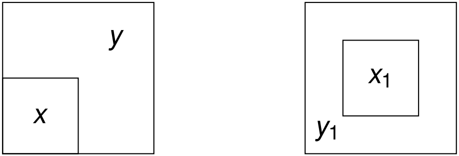
Definition (of a relation of non-tangential inclusion)

$$x \ll y \overset{\text{df}}{\iff} \forall_{z \in M}(z \wr y \implies z \;)(\; x)\,. \qquad (\text{df} \ll)$$

If $x \ll y$ we say that x is non-tangentially included in y.

Example

Region x touches the complement of y, but x_1 is non-tangentially included in y_1, that is $x_1 \ll y_1$.



Non-tangential inclusion – elementary properties

$$x \ll y \overset{\text{df}}{\Longleftrightarrow} \forall_{z \in M}\big(z \wr y \Longrightarrow z \mathrel{)(} x\big)\,. \tag{df $\ll$}$$

Lemma

(i) $x \ll y \iff y = \mathbf{1} \vee \big(y \neq \mathbf{1} \wedge x \mathrel{)(} -y\big)$,

(ii) $y = \mathbf{1} \vee \big(y \neq \mathbf{1} \wedge x \mathrel{)(} -y\big) \Longrightarrow x \sqsubseteq y$,

(iii) $x \ll y \Longrightarrow x \sqsubseteq y$,

(iv) $\forall_{x \in R} x \ll \mathbf{1}$, so $\mathbf{1} \ll \mathbf{1}$.

The connection relation

$$x \mathbin{\mathbf{C}} y \overset{\text{df}}{\iff} \neg x \mathrel{)(} y. \tag{def $\mathbf{C}$}$$

Since $\mathbf{C}$ is a complement of $\mathrel{)(}$ one can easily express counterparts of the axioms (S1)–(S3) by means of the connection relation.

$$\forall_{x,y\in\mathbb{R}}(x \sqsubseteq y \Longrightarrow x \mathbin{\mathbf{C}} y)\,, \tag{S1$'$}$$

$$\forall_{x,y\in\mathbb{R}}(x \mathbin{\mathbf{C}} y \Longrightarrow y \mathbin{\mathbf{C}} x)\,, \tag{S2$'$}$$

$$\forall_{x,y,z\in\mathbb{R}}(x \sqsubseteq y \wedge z \mathbin{\mathbf{C}} x \Longrightarrow z \mathbin{\mathbf{C}} y)\,. \tag{S3$'$}$$

The axioms above are counterparts of some of the axioms of the so called connection structures.

Representatives of points

Definition (of a representative of a point)

We say that $X \in \mathcal{P}(\mathbb{R})$ is a representative of a point iff X satisfies the following three conditions:

$$\forall_{u,v \in X} \big(u \neq v \Longrightarrow u \ll v \vee v \ll u\big)\,, \tag{G1}$$

$$\forall_{u \in X} \exists_{v \in X}\ v \ll u\,, \tag{G2}$$

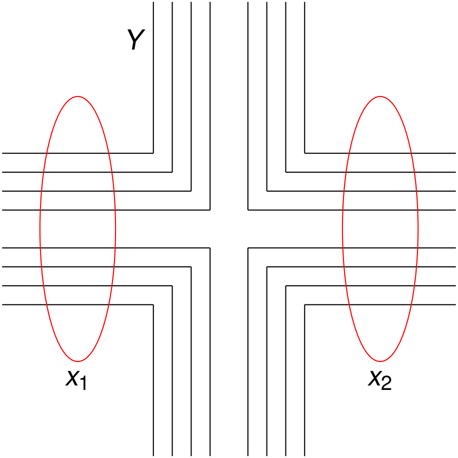
$$\forall_{u,v \in \mathbb{R}} \Big(\forall_{z \in X} \big(z \bigcirc u \wedge z \bigcirc v\big) \implies u \,\mathbf{C}\, v\Big)\,. \tag{G3}$$

Let **Q** be the family of all representatives of points:

$$\mathbf{Q} := \{X \in \mathcal{P}(\mathbb{R}) \mid X \neq \emptyset \wedge X \text{ satisfies (G1), (G2) and (G3)}\}\,. \tag{df $\mathbf{Q}$}$$

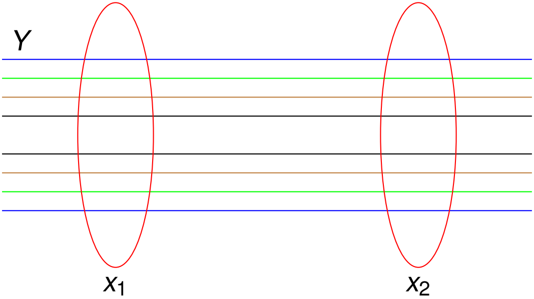
Representatives of points

Example



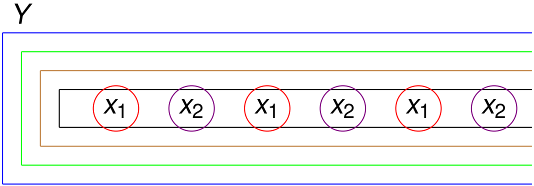
Representatives of points

Example



Representatives of points

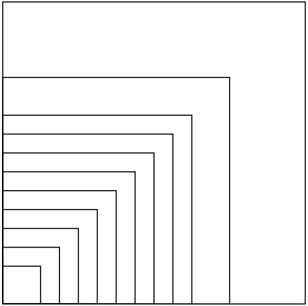
Example



Representatives of points

Example

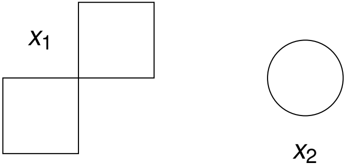
QUESTION: Suppose the whole slide is the space. Does the set of regions below satisfy conditions of a representative of a point?



Representatives of points

Definition

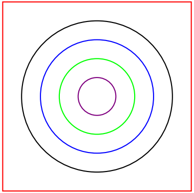
A region $x \in \mathbb{R}$ is coherent iff for all $y_1, y_2 \in \mathbb{R}$ such that $x = y_1 \sqcup y_2$, it is the case that $y_1 \; \mathbf{C} \; y_2$ (i.e. $\neg y_1 \;)(\; y_2$).



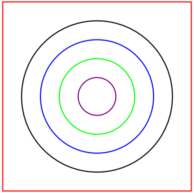
Both regions x_1 and x_2 are coherent, but their sum $x_1 \sqcup x_2$ is not.

Representatives of points

Example



x_1



x_2

Grzegorczyk's axiom

$$x \mathbf{C} y \Longrightarrow \exists_{Q \in \mathbf{Q}} \Big(\exists_{z \in Q} (x \bigcirc y \Longrightarrow z \sqsubseteq x \sqcap y) \wedge \forall_{z \in Q} (z \bigcirc x \wedge z \bigcirc y) \Big) . \quad \text{(G)}$$

Any structure $\langle \mathbb{R}, \sqsubseteq,)(\rangle$ satisfying axioms (S0)–(S4),(G) will be called Grzegorczyk's structure or G-structure.

Fact

$\mathbf{Q}$ *is not empty.*

Proof.

It is the case that **1 C 1**. That means that there exists a set $X \in \mathbf{Q}$ satisfying all conditions listed in (G). $\square$

Fact

$\forall_{x \in \mathbb{R}} \exists_{Q \in \mathbf{Q}} \Big(\forall_{z \in Q} \; z \bigcirc x \wedge \exists_{z \in Q} \; z \sqsubseteq x \Big).$

Proof.

From the reflexiveness of **C** and $\sqsubseteq$. $\square$

Points in G-structures

Definition (of a filter)

A filter in a mereological structure $\langle M, \sqsubseteq \rangle$ is any non-empty set $F \subseteq M$ such that

(f1) if $x, y \in F$, then $x \bigcirc y$ and $x \sqcap y \in F$,

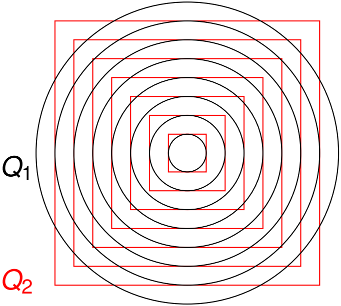
(f2) if $x \in F$ and $x \sqsubseteq y$, then $y \in F$.

Definition (of a point)

By a point in any G-structure we will mean any filter in $\langle \mathbb{R}, \sqsubseteq \rangle$ generated by some element of a family $\mathbf{Q}$. Let us denote the set of all points by 'Π'; for any set β from $\mathcal{P}(\mathbb{R})$:

$$\beta \in \Pi \iff \exists_{Q \in \mathbf{Q}}\, \beta = \{\, x \in \mathbb{R} : \exists_{y \in Q}\ y \sqsubseteq x \,\}. \qquad (\mathrm{df}\ \Pi)$$

Definition of a point – short explanation



Q_1 and Q_2 represent the same point.

Existence of points

Fact

Since $\mathbf{Q} \neq \emptyset$*, then from* (df $\mathbf{Q}$), (df Π) *and properties of* $\mathbf{1}$ *it follows that:*

$$\Pi \neq \emptyset \wedge \forall_{\alpha \in \Pi}\, \mathbf{1} \in \alpha\,. \tag{1}$$

Fact

For any $x \in \mathbb{R}$ *and* $\alpha \in \Pi$ *we have that:*

$$\begin{aligned} x \in \alpha &\iff \exists_{y \in \alpha}\, y \sqsubseteq x \\ &\iff \exists_{y \in \alpha}\, y \ll x\,. \end{aligned} \tag{2}$$

Basic properties of points

Theorem

The following conditions are consequences of axioms (S0)–(S3),(G)*:*

$$\forall_{x \in \mathbb{R}} \, \exists_{\alpha \in \Pi} \; x \in \alpha \,, \tag{3}$$

$$\forall_{x,y \in \mathbb{R}} \Big(x \bigcirc y \implies \exists_{\alpha \in \Pi} \; x \sqcap y, x, y \in \alpha \Big) \,, \tag{4}$$

$$\forall_{x,y \in \mathbb{R}} \forall_{\alpha \in \Pi} \Big(x, y \in \alpha \implies x \bigcirc y \wedge x \sqcap y \in \alpha \Big) \,, \tag{5}$$

$$\forall_{x,y \in \mathbb{R}} \Big(x \mathrel{\mathbf{C}} y \iff \exists_{\alpha \in \Pi} \forall_{z \in \alpha} (z \bigcirc x \wedge z \bigcirc y) \Big) \,, \tag{6}$$

$$\forall_{x \in \mathbb{R} \setminus \{\mathbf{1}\}} \Big(x \mathrel{\mathbf{C}} -x \iff \exists_{\alpha \in \Pi} \; x \notin \alpha \wedge -x \notin \alpha \Big) . \tag{7}$$

Definition of the operation **Irl**

On the set $\mathbb{R}$ we introduce an operation that ascribes to an arbitrary region x the set of all points α such that $x \in \alpha$.

Formally we define the operation **Irl**: $\mathbb{R} \to \mathcal{P}_{+}(\Pi)$ such that:

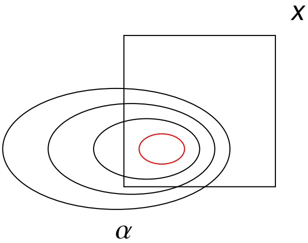
$$\mathbf{Irl}(x) := \{\, \alpha \in \Pi \mid x \in \alpha \,\}\,. \tag{df Irl}$$

For a region x we will call **Irl**(x) the set of internal points of x.

Fact

$$\begin{aligned} \alpha \in \mathbf{Irl}(x) &\iff \exists_{y \in \alpha}\, y \sqsubseteq x \\ &\iff \exists_{y \in \alpha}\, y \ll x\,. \end{aligned}$$

Definition of **lrl** – an example

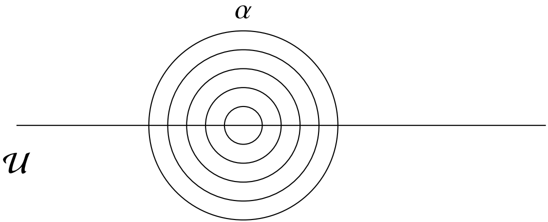


Open sets

Definition (of an open set)

$$\mathbf{O} = \{\, \mathcal{U} \in \mathcal{P}(\Pi) \mid \forall_{\alpha \in \mathcal{U}}\, \exists_{x \in \alpha}\, \mathbf{Irl}(x) \subseteq \mathcal{U} \,\}. \qquad \text{(df } \mathbf{O}\text{)}$$

Example (negative)



Open sets

Definition

$$\mathbf{B} := \{\mathbf{Irl}(x) \mid x \in \mathbb{R}\}\,. \tag{df $\mathbf{B}$}$$

Fact

(a) $\mathbf{B} \subseteq \mathbf{O}$*, that is for every* $x \in \mathbb{R}$ *we have* $\mathbf{Irl}(x) \in \mathbf{O}$*;*

(b) $\forall_{\mathcal{U}\in\mathbf{O}}\forall_{\alpha\in\mathcal{U}}\,\exists_{\mathcal{V}\in\mathbf{B}}\ \alpha \in \mathcal{V} \subseteq \mathcal{U}$*;*

(c) $\mathbf{O} = \{\,\mathcal{A} \in \mathcal{P}(\Pi) \mid \exists_{\mathcal{F}\subseteq\mathbf{B}}\ \mathcal{A} = \bigcup\mathcal{F}\,\}$*;*

(d) $\mathbf{B}$ *is a basis of* $\langle\Pi, \mathbf{O}\rangle$*.*

Open sets – an alternative way

Definition

A family of sets $\mathcal{B} \subseteq \mathcal{P}(X)$ is a basis for the set X iff

- $\bigcup \mathcal{B} = X$,
- if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

- Prove the following facts:

$$\mathbf{Irl}(\mathbf{1}) = \Pi = \textstyle\bigcup_{x \in \mathbb{R}} \mathbf{Irl}(x)\,,$$

$$x \bigcirc y \overset{\text{df}}{\iff} \mathbf{Irl}(x) \cap \mathbf{Irl}(y) \neq \emptyset\,,$$

$$x \bigcirc y \implies \mathbf{Irl}(x) \cap \mathbf{Irl}(y) = \mathbf{Irl}(x \sqcap y)\,.$$

- It follows that **B** is a basis.
- Introduce **O** in the standard way.

$\langle \Pi, \mathbf{O} \rangle$ is a Hausdorff space

Lemma

$\forall_{\beta,\gamma\in\Pi}\Big(\beta \neq \gamma \implies \exists_{x\in\beta}\,\exists_{y\in\gamma}\; x \wr y\Big).$

Corrolary

Topological space $\langle \Pi, \mathbf{O} \rangle$ is a Hausdorff space, that is it satisfies (T_2).

Proof.

- Let $\beta \neq \gamma$.
- Thus there are regions $x \in \beta$ and $y \in \alpha$ such that $x \wr y$.
- Consider the sets $\mathbf{Irl}(x)$ and $\mathbf{Irl}(y)$.

$\square$

The End
of
Part IV